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Numerical Solutions of Volterra Integral Equations Through Galerkin Weighted Residual Method with Charlier Polynomials

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ABSTRACT

Volterra Integral Equations (VIEs) are a significant class of integral equations with broad applications in various fields, such as mathematical physics, engineering, biology, economics, and more. In this paper, we numerically solve the linear VIEs of both the first and second kind, with both regular and singular kernels, using the Galerkin Weighted Residual Method. Actually, we derive a straightforward and efficient matrix formulation by the Galerkin Method for each type of VIE, employing piecewise Charlier polynomials as the basis functions in the trial solution. Several numerical examples are tested to verify the effectiveness of the proposed method. The numerical results obtained by the proposed method converge monotonically to the exact solutions and, in some cases, achieve the exact solution. In addition, the proposed Charlier polynomials-based Galerkin method significantly outperforms other state-of-the-art methods for the numerical solution of VIEs.

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Introduction

Volterra Integral Equations (VIEs) play an important role in many fields, including biology, mathematics, physics, population dynamics, engineering, and more (Lonseth, 1977; Jerri, 1999). They have also been extensively used to model complex dynamical systems (Guenther et al., 1996; Elfelsoufi et al., 2016). Because of their broad spectrum of applications in both theoretical and applied fields, they attract the interest of many researchers, making the study of VIEs important. Hence, finding an appropriate solution to VIEs is crucial for addressing the related fields where they may be applied. Several numerical and analytical approaches have been developed for solving VIEs over the past few years (Saran et al., 2000; Nohel, 1962).

Several mathematical attempts have been proposed and developed over the last few years to address the numerical solution of VIEs. Islam and Rahman solved the VIEs of the first and second kind, as well as nonlinear VIEs, using the Galerkin Weighted Residual method (Islam & Rahman, 2013). They independently adopted Hermite and Chebyshev piecewise polynomials as the basis functions in the Galerkin method. They tested their approach on several numerical problems and established a comparison between Hermite and Chebyshev polynomial-based approximations. Their approximate solutions demonstrated that the Chebyshev polynomials-based approximation performs better than Hermite polynomials-based approximation. Rahman et al. solved the VIEs of first and second kind using the Galerkin Weighted Residual method, considering Laguerre piecewise polynomials as the basis functions (Rahman et al., 2012). They developed an effective matrix formulation through the Galerkin method to obtain approximate results. Their numerical results are better than those in other polynomials-based research works. Zarnan and Hameed developed a Bernstein polynomials-based approximation method for solving VIEs (Zarnan & Hameed, 2024). They derived a simple and efficient matrix formulation with the help of Bernstein polynomials for the numerical solutions of VIEs. They tested several numerical examples to demonstrate the performance of their method. Their numerical results are excellent and superior to the existing methods available in the literature.

Maleknejad et al. applied Bernstein's approximation to solve VIEs of the first kind, the second kind, as well as singular types of these equations (Maleknejad et al., 2011). Actually, they approximated the unknown function using Bernstein's approximation for VIEs. They tested several numerical examples to verify their proposed method and proved that their numerical results have high accuracy, and the method is very effective, simple and reliable.

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Berenguer et al. developed a new approach for the numerical resolution of VIEs of the second kind with the help of the Schauder bases and Geometric Series theorem in Banach space (Berenguer et al., 2009). Using the proposed method, they solved several numerical examples to verify its performance. Their computed results surpass those of other standard methods. For solving the Able's integral equation of the second kind, Shahsavaran developed a numerical approach using the Block-Pulse Function (BPF) and Taylor expansion by the collocation method (Shahsavaran, 2011). He used a k sets of BPF to obtain the numerical results. He adopted several numerical examples to justify the method's performance. This paper compared the numerical results with those obtained using the Legendre wavelet method. The author concluded that the obtained approximate results have good accuracy compared to the results obtained using the Legendre wavelet method.

To solve VIEs of the first kind with a weakly singular kernel, Kamoh et al. proposed a Hermite piecewise polynomial-based Galerkin method (Kamoh et al., 2022). They formulated and solved a system of algebraic equations through Hermite polynomials for the numerical solutions. Several numerical examples were tested to verify the validity and efficiency of the method's performance. The numerical results obtained from these examples demonstrated that the approximate solutions usually coincide with the exact solutions after using only a few terms of the polynomial basis functions. Shahsavaran developed a computational method for VIEs of the second kind with a weakly singular kernel using Haar wavelets and the properties of Block-Pulse Functions (BPF) (Shahsavaran, 2011). To measure the proficiency and accuracy of the method, several numerical examples were considered. The author concluded that the proposed Haar wavelet method shows high-order convergence, and is simple, and requires fewer computations compared to other approximation approaches.

In this paper, we attempt to find the numerical solution of VIEs of both the first and second kind with regular and singular kernels. To achieve this, a simple and efficient matrix formulation is developed using the Galerkin Method. Specifically, the well-known Charlier piecewise polynomials are adopted as the basis functions in the trial solution of the Galerkin method. Several numerical examples are considered and tested to evaluate the performance of the proposed Charlier polynomial-based Galerkin method. This paper is organized as follows: In the section "Charlier Polynomial", the Charlier piecewise polynomials are defined mathematically and presented graphically. In the section "Matrix Formulation", we derive a matrix formulation using the Galerkin Weighted Residual method, considering the Charlier polynomials as the basis functions. The formulation for both the first and second kinds of VIEs are described here. In the section "Numerical Examples", several numerical examples are tested using our proposed Charlier polynomials-based Galerkin approximation method. For the tested problems, the approximate results and absolute errors are calculated and discussed here. Performance comparisons with other standard methods are also included in this section. The final section, "Conclusion", contains the concluding remarks of this paper.

Charlier Polynomial

The Charlier Polynomial is a well-known orthogonal polynomial, which can be defined of order n as follows (Peccati & Taqqu, 2011):

$$C_n(x,a)=\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\frac{x^r}{a^r} \tag{1}$$

In this case, the first few Charlier polynomials are as follows:

$$C_0(x,a)=1$$

$$C_1(x,a)=\frac{x}{a}-1$$

$$C_2(x,a)=\frac{x^2}{a^2}-2\frac{x}{a}+1$$

$$C_3(x,a)=\frac{x^3}{a^3}-3\frac{x^2}{a^2}+3\frac{x}{a}-1$$

For simplicity of calculation, we have taken $a=1$. Then, the Charlier polynomial of order n can be rewrite as follows:

$$C_n(x,a)=\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}x^r \tag{2}$$

In this case, the first few Charlier polynomials are as follows:

$$C_0(x,1)=1$$

$$C_1(x,1)=x-1$$

$$C_2(x,1)=x^2-2x+1$$

$$C_3(x,1)=x^3-3x^2+3x-1$$

$$C_4(x,1)=x^4-4x^3+6x^2-4x-1$$

$$C_5(x,1)=x^5-5x^4+10x^3-10x^2+5x-1$$

The first five Charlier piecewise polynomials over the interval $[-3,\ 6]$ by taking $a=1$ are generated and demonstrated in Figure 1.

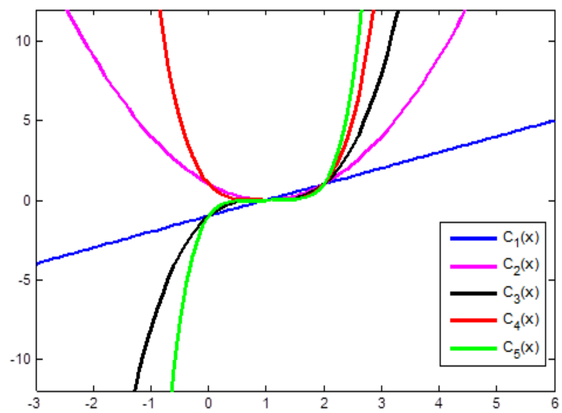

Figure 1. First five Charlier polynomials over the interval $[-3,\ 6]$ with $a=1$.

Matrix Formulation

In this section, we derive a system of linear equations in matrix form for solving Volterra Integral Equations

(VIEs). The Galerkin Weighted Residual method is used as the numerical technique, with Charlier polynomials utilized as the basis functions in the trial solution. VIEs of both the first and second kind are mathematically formulated for numerical approximation via Galerkin Weighted Residual approach. The following subsections contain the detailed procedure for the matrix formulation of each type of integral equation.

Matrix Formulation of First Kind Volterra Integral Equation

The 1st kind Volterra Integral Equation (VIE) is a type of integral equation named after the Italian mathematician Vito Volterra. This equation is fundamental in the study of various problems in mathematical physics, biology, and other applied sciences. The standard from of FKVIE can be written as follows (Jerri, 1999):

$$\int_a^x k(x,t)\phi(t)\,dt = f(x), \qquad a \le x \le b \tag{3}$$

where $k(x,t)$ is a known kernel function that depends on both x and t, $\phi(x)$ is the unknown function that we need to determine, and $f(x)$ is a known function that satisfies $f(a)=0$. In this case, the Galerkin is used to find an approximate solution $\tilde{\phi}(x)$ by assuming the trial solution as follows (Islam & Rahman, 2013):

$$\tilde{\phi}(x) = \sum_{i=0}^{n} a_i C_i(x) \tag{4}$$

where $C_i(x)$ is the Charlier polynomial of degree i, n is the number of Charlier polynomials, and a_i represents the unknown co-efficient that needs to be determined. Now by substituting the series $\tilde{\phi}(x)$ into the equation (3), we get:

$$\sum_{i=0}^{n} a_i \int_a^x k(x,t) C_i(t)\,dt = f(x), a \le x \le b \tag{5}$$

Now, the Galerkin equations can be obtained by multiplying on both sides of equation (5) by the weighted functions $C_j(x)$ and then integrating the resulting expression with respect to x over the domain $[a,b]$. This process yields the following equations:

$$\sum_{i=0}^{n} a_i \int_a^b \left[\int_a^x k(x,t) C_i(t)\,dt\right] C_j(x)\,dx = \int_a^b C_j(x) f(x)\,dx$$
$$i, j = 0,1,2,3,...n. \tag{6}$$

The left-hand side of equation (6) consists of two integrals- an inner integral and an outer integral. The inner integral is a function of t only. Since the limits of integration are from a to x, the value of this integration will be in terms of x. The outer integral, which depends on x, will yield a constant value after being integrated from a to b. On the right-hand side of equation (6), the integration limits are from a to b, resulting in a constant value. For each $j = 0, 1, 2, 3, \cdots, n$ a system of linear equations with $n+1$ unknown will be found as a_i with $i = 0, 1, 2, 3, \cdots, n$. This leads to the generation of a system of linear equations with $n+1$ unknowns and $n+1$ linear equations. Therefore, the system of linear equations can be written in matrix form as follows:

$$\sum_{i=0}^{n} a_i K_{i,j} = F_j, \qquad i, j = 0, 1, 2, 3, \cdots, n \tag{7}$$

where $K_{i,j}$ is the coefficient matrix and F_j is the constant matrix, both of which can be formed as follows:

$$K_{i,j} = \int_a^b \left[\int_a^x k(x,t) C_i(t)\,dt\right] C_j(x)\,dx, \; i, j = 0,1,2,3,\cdots n \tag{8}$$

$$F_j = \int_a^b C_j(x) f(x)\,dx, \quad j = 0,1,2,3,\cdots,n \tag{9}$$

By solving the system of equations (7), the unknown coefficients a_i can be determined. Substituting these coefficients a_i into equation (4), the approximate solution $\tilde{\phi}(x)$ for 1st kind VIE can be obtained.

Matrix Formulation of Second Kind Volterra Integral Equation

The 2nd kind Volterra Integral Equation (VIE) is another important type of integral equation named after Vito Volterra. It plays a crucial role in various fields of applied mathematics, physics, and engineering. The general form of the SKVIE can be written as follows (Jerri, 1999):

$$c(x)\phi(x) + \lambda \int_a^x k(x,t)\phi(t)\,dt = f(x), x \in [a,b] \tag{10}$$

where $\phi(x)$ denotes the unknown function that we seek to determine, $f(x)$ represents a known function (sometimes called the inhomogeneous term or the free term), $k(x,t)$ indicates a known kernel function, which depends on both x and t, $c(x)$ is a given function, and λ is a constant. Let the approximate solution of equation (10) be $\tilde{\phi}(x)$, which can be obtained based on equation (11).

$$\tilde{\phi}(x) = \sum_{i=0}^{n} a_i C_i(x) \tag{11}$$

In equation (11), $C_i(x)$ is the Charlier polynomial of degree i, and n is the number of Charlier polynomials. Here, a_i represents the unknown coefficient that needs to be determined. By applying the same procedure used in the formulation of the 1st kind VIE above, we get a system of linear equations in matrix form as follows:

$$\sum_{i=0}^{n} a_i K_{i,j} = F_j, \quad i, j = 0,1,2,3,\cdots,n \tag{12}$$

where coefficient matrix and constant matrix are obtained with the help of the equations (13) and (14), respectively.

$$K_{i,j} = \int_a^b \left[c(x) C_i(x) + \lambda \int_a^x k(x,t) C_i(t)\,dt\right] C_j(x)\,dx,$$
$$i, j = 0,1,2,3,\cdots,n \tag{13}$$

$$F_j = \int_a^b C_j(x) f(x)\,dx, \quad j = 0,1,2,3,\cdots,n \tag{14}$$

By solving the system of equations (12), the unknown coefficients a_i can be determined. Substituting these coefficients a_i into the equation (11), the approximate solution $\tilde{\phi}(x)$ for SKVIE in terms of Charlier polynomials can be found.

Numerical Examples

In this section, several numerical examples involving first- and second-kind Volterra integral equations with regular and singular kernels are tested to evaluate our proposed Charlier polynomials-based Galerkin method. The numerical examples are taken from the previously published research works. Comparisons between the results of our proposed method and the existing standard methods are also presented in this section. Actually, the obtained results are compared with those provided in the literature. The program for technical computations is coded and executed on the MATLAB 13Ra platform. The performance indicator, absolute error, can be computed based on the equation (15). If $\phi(x)$ is the exact solution and $\tilde{\phi}(x)$ is the approximate solution, the absolute error can be calculated as follows:

$$\text{Absolute Error} = \left|\phi(x)-\tilde{\phi}(x)\right| \qquad (15)$$

Example 1: Consider 1st kind Abel's Integral Equation, as given in (Rahman et al., 2012):

$$\int_0^x \frac{1}{\sqrt{x-t}}\phi(t)\,dt = \frac{2}{105}\sqrt{x}\left(105-56x^2+48x^3\right); \quad \text{where } x\in[0,1] \qquad (16)$$

The exact solution of equation (16) is $\phi(x)=x^3-x^2+1$. Using Charlier polynomial as the basis functions, the approximate solution is found to be $\tilde{\phi}(x)=x^3-x^2+1$ for $n\geq 3$, which coincides with the exact solution. For this problem, Rahman et al. found the error to be of the order of 10^{-16} using Laguerre polynomials as the trial function for $n=10$ (Rahman et al., 2012). Islam and Rahman obtained the approximate solution $\tilde{\phi}(x)=x^3-x^2+1$ using the Chebyshev polynomials as the trial functions for $n\geq 3$, and using the Hermite polynomials as the trial functions, they found the error to be of the order of 10^{-16} for $n=10$ (Islam & Rahman, 2013). Kamoh et al. approximated the solution using Hermite polynomials as the trial functions for different polynomial orders (Kamoh et al., 2022). On the other hand, Maleknejad et al. found the solution through Bernstein approximation for various polynomial orders, with the error being of the order of 10^{-7} for $n=10$ (Maleknejad et al., 2011). The numerical results and comparisons with previous studies are shown in Table 1.

Example 2: Consider the VIE of the 1st kind, as described in (Islam & Rahman, 2013):

$$\int_0^x (5+3x-3t)u(t)\,dt = 5x^2+x^3 \text{ where } x\in[0,1] \qquad (17)$$

The exact solution provided for this integral equation (17) is $u(x)=2x$. Using Charlier polynomials as the basis functions, we get the approximate solution $\tilde{u}(x)=2x$ for $n\geq 1$, which coincides with the exact solution. On the contrary, Islam and Rahman found the numerical solution, which is the same as the exact solution, using both Hermite and Chebyshev polynomials for $n\geq 1$ (Islam & Rahman, 2013).

Example 3: Assume 1st kind Abel's Integral Equation, as shown in (Islam & Rahman, 2013):

$$\int_0^x \frac{1}{\sqrt{x-t}}u(t)\,dt = x^r \qquad x\in[0,1] \qquad (18)$$

For $r=\frac{3}{2}$, the exact solution of this first-kind equation is $u(x)=\frac{3}{4}x$. For $n\geq 1$, using Charlier polynomials as the basis functions, we obtain the numerical solution $\tilde{u}(x)=\frac{3}{4}x$, which matches the analytical solution. Islam and Rahman obtained the same results as we did, but they used Hermite and Chebyshev polynomials independently in case of $n\geq 1$ (Islam & Rahman, 2013).

Example 4: Consider the weakly singular VIE of 2nd kind, as presented in (Rahman et al., 2012):

$$\phi(x)-\int_0^x \frac{1}{\sqrt{x-t}}\phi(t)\,dt = x^7\left(1-\frac{4096}{6435}\sqrt{x}\right); x\in[0,1] \qquad (19)$$

The exact solution of this second-kind equation (19) is $\phi(x)=x^7$. Using Charlier polynomial as the basis functions, the approximate result for $n\geq 7$ is found to be $\tilde{\phi}(x)=x^7$, which refers to the exact solution. For this problem, Rahman et al. found the error to be up to the order of 10^{-11} using Laguerre polynomials as the basis functions for $n=10$ (Rahman et al., 2012). Islam and Rahman found the error to be of the order of 10^{-15} using Hermite polynomial as trial functions for $n=10$, and using Chebyshev polynomial, the approximate solution was found as $\tilde{\phi}(x)=x^7$ for $n\geq 7$ refers to the exact solution (Islam & Rahman, 2013). Shahsavaran approximated the solution using the Haar wavelet as the approximation function (Shahsavaran, 2011). The numerical results and comparisons with others are demonstrated in Table 2.

Example 5: Let the VIE of 2nd kind, as outlined in (Rahman et al., 2012):

$$\phi(x)+\int_0^x \frac{1}{\sqrt{x-t}}\phi(t)\,dt = x^2+\left(\frac{16}{15}\right)x^{\frac{5}{2}}, x\in[0,1] \qquad (20)$$

The analytical solution of this second-kind equation (20) is $\phi(x)=x^2$. Using Charlier polynomials as basis function, the approximate result for $n\geq 2$ is found to be $\tilde{\phi}(x)=x^2$, which is the exact solution of the problem. Using Laguerre polynomial as the trial function, Rahman et al. obtained the exact solution for $n\geq 2$ (Rahman et al., 2012). Islam and Rahman found the exact solution using Hermite and Chebyshev polynomials independently for $n\geq 2$ (Islam & Rahman, 2013). On the other hand, using the Block Pulse Function and Taylor expansion, Shahsavaran computed the error to be of the order 10^{-3} (Shahsavaran, 2011). The numerical results and comparisons with other methods for this problem is shown in Table 3.

Table 1: Approximate Results and Comparisons with others for Example 1

		Charlier Polynomial **(Proposed)** $n \ge 3$	Laguerre Polynomial (Rahman et al., 2012) $n = 10$	Hermite Polynomial (Islam & Rahman, 2013) $n = 10$	Chebyshev Polynomial (Islam & Rahman, 2013) $n \ge 3$	Hermite Polynomial (Kamoh et al., 2022) $n \ge 3$
x	**Exact Solution**	**Absolute Error**	**Absolute Error**	**Absolute Error**	**Absolute Error**	**Absolute Error**
0.0	1.0000000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.9910000000	0.0000000	3.3306691E-016	0.0000000	0.0000000	-------------------
0.2	0.9680000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.3	0.9370000000	0.0000000	5.5511151E-016	0.0000000	0.0000000	-------------------
0.4	0.9040000000	0.0000000	7.7715612E-016	1.1102230E-016	0.0000000	0.0000000
0.5	0.8750000000	0.0000000	0.0000000	0.0000000	0.0000000	-------------------
0.6	0.8560000000	0.0000000	1.6653345E-015	1.1102230E-016	0.0000000	0.0000000
0.7	0.8530000000	0.0000000	1.5543122E-015	0.0000000	0.0000000	-------------------
0.8	0.8720000000	0.0000000	1.1102230E-016	1.1102230E-016	0.0000000	0.0000000
0.9	0.9190000000	0.0000000	4.4408921E-016	0.0000000	0.0000000	---------------------
1.0	1.0000000000	0.0000000	1.7763568E-015	0.0000000	0.0000000	0.0000000

Example 6: Assume the VIE of 2nd kind, as described in (Islam & Rahman, 2013):

$$u(x) - \int_0^x txu(t)\,dt = x^5 - \left(\frac{x^8}{7}\right); \qquad x \in [0,1] \qquad (21)$$

The exact solution of this second-kind problem is $u(x) = x^5$. Using Charlier polynomials as basis function, the approximate result is found to be $\tilde{u}(x) = x^5$ by taking $n \ge 5$, which refers to the exact solution. Islam and Rahman captured the exact solution using Hermite and Chebyshev polynomials independently for $n \ge 5$ (Islam & Rahman, 2013). Using the Schauder bases and geometric series theorem in Banach space, Berenguer et al. found an approximate solution for the problem, with an error in order up to 10^{-5} (Berenguer et al., 2009).

Example 7: Consider the VIE of 2nd kind (Zarnan & Hameed, 2024):

$$u(x) = e^x + \int_0^x u(t)\,dt, \quad 0 \le x \le 1 \qquad (22)$$

The exact solution of this second-kind integral equation (22) is $u(x) = e^x(1+x)$. Using Charlier polynomial as basis function, we found an approximate result with an error of order 10^{-22} by taking $n = 15$. Zarnan et al. used the Bernstein approximation to solve the problem, obtaining an approximate solution for $n = 3$ with an error of the order 10^{-4} (Zarnan & Hameed, 2024). The results and comparison between Charlier polynomial and Bernstein approximation method are shown in Table 4. Figure 2. illustrates the performance comparison between the exact and numerical solutions for this problem.

Example 8: Assume the VIE of 2nd kind (Zarnan & Hameed, 2024):

$$u(x) = x + \int_0^x (t-x)u(t)\,dt, x \in [0,1] \qquad (23)$$

The exact solution of this integral equation (23) is $u(x) = \sin(x)$.Using Charlier polynomials as the basis function, we approximated the solution for $n = 10$. The error in this case is of the order 10^{-17}. Zarnan et al. used the Bernstein polynomial approach to solve the problem, approximating the solution for $n = 10$ with an error of the order 10^{-13} (Zarnan & Hameed, 2024). The results and comparison between the Charlier and Bernstein polynomial methods are shown in Table 5, while Figure 3.

Table 2: Approximate Results and Comparisons with others for Example 4

		Charlier polynomial **(Proposed)** $n \ge 7$	Laguerre Polynomial (Rahman et al., 2012) $n = 10$	Hermite Polynomial (Islam & Rahman, 2013) $n = 10$	Chebyshev Polynomial (Islam & Rahman, 2013) $n \ge 7$	Haar Wavelet Function (Shahsavaran, 2011)
x	**Exact Solution**	**Absolute Error**	**Absolute Error**	**Absolute Error**	**Absolute Error**	**Absolute Error**
0.0	0.0000000000	0.0000000	0.0000000	0.0000000	0.0000000	0.00000
0.1	0.0000001000	0.0000000	1.3790466E-11	1.0518363E-15	0.0000000	0.00000
0.2	0.0000128000	0.0000000	1.5311828E-11	5.2011042E-16	0.0000000	0.00000
0.3	0.0002187000	0.0000000	8.1805515E-12	1.5883562E-17	0.0000000	0.00003
0.4	0.0016384000	0.0000000	2.4509244E-11	2.1811979E-15	0.0000000	0.00003
0.5	0.0078125000	0.0000000	1.4551915E-11	0.0000000	0.0000000	0.00096
0.6	0.0279936000	0.0000000	4.6102080E-11	5.9674488E-16	0.0000000	0.00071
0.7	0.0823543000	0.0000000	1.8066840E-10	2.4980018E-16	0.0000000	0.00328
0.8	0.2097152000	0.0000000	5.3086729E-11	3.4694470E-15	0.0000000	0.01006
0.9	0.4782969000	0.0000000	8.3868856E-11	4.1633363E-15	0.0000000	0.00307
1.0	1.0000000000	0.0000000	4.3655746E-11	3.5527137E-15	0.0000000	---------

displays the comparison between the exact and numerical solutions for this problem.

Table 3: Approximate Results and Comparisons with others for Example 5

		Charlier polynomial **(Proposed)** $n \geq 2$	Lagurrer Polynomial (Rahman et al., 2012) $n \geq 2$	Hermit Polynomial (Islam & Rahman, 2013) $n \geq 2$	Chebeshev Polynomial (Islam & Rahman, 2013) $n \geq 2$	Block Pulse Function (Shahsavaran, 2011)
x	**Exact Solution**	**Absolute Error**	**Absolute Error**	**Absolute Error**	**Absolute Error**	**Absolute Error**
0.0	0.00000000000	0.0000000	0.0000000	0.0000000	0.0000000	----------
0.1	0.01000000000	0.0000000	0.0000000	0.0000000	0.0000000	1×10^{-3}
0.2	0.04000000000	0.0000000	0.0000000	0.0000000	0.0000000	1×10^{-3}
0.3	0.09000000000	0.0000000	0.0000000	0.0000000	0.0000000	2×10^{-3}
0.4	0.16000000000	0.0000000	0.0000000	0.0000000	0.0000000	7×10^{-3}
0.5	0.25000000000	0.0000000	0.0000000	0.0000000	0.0000000	1×10^{-2}
0.6	0.36000000000	0.0000000	0.0000000	0.0000000	0.0000000	1×10^{-2}
0.7	0.49000000000	0.0000000	0.0000000	0.0000000	0.0000000	3×10^{-3}
0.8	0.64000000000	0.0000000	0.0000000	0.0000000	0.0000000	5×10^{-3}
0.9	0.81000000000	0.0000000	0.0000000	0.0000000	0.0000000	1×10^{-2}
1.0	1.00000000000	0.0000000	0.0000000	0.0000000	0.0000000	-----------

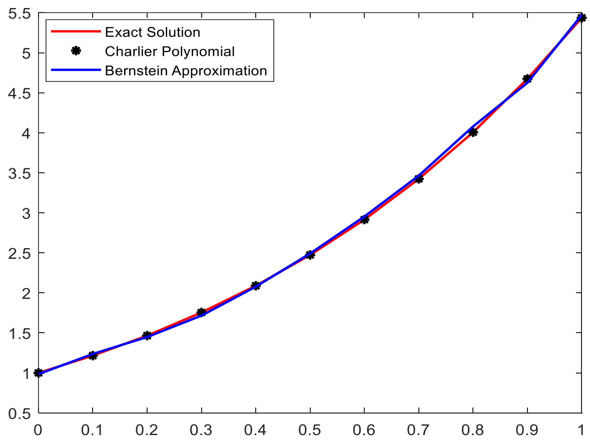

Figure 2. Exact and Approximate solutions of Example 7

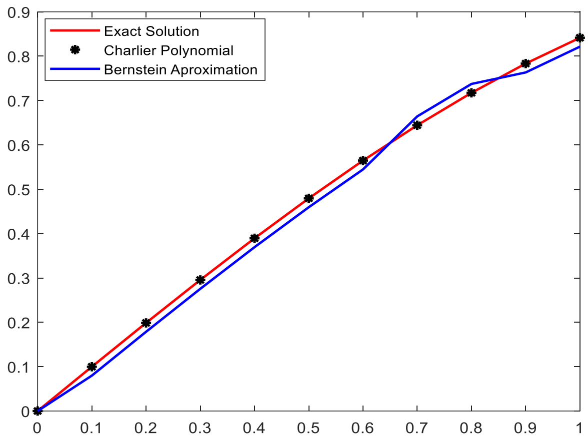

Figure 3. Exact and Approximate solutions of Example 8

Table 4: Approximate Results and Comparison for Example 7

		Charlier polynomial **(Proposed)** $n = 15$		Bernstein Polynomial (Zarnan & Hameed, 2024) $n = 3$
x	**Exact solution**	**Approximate**	**Absolute Error**	**Absolute Error**
0.0	1.00000000000	1.00000000000	2.3024816E-21	5.4670224E-003
0.1	1.21568800988	1.21568800988	5.4255035E-22	9.0893339E-004
0.2	1.46568330979	1.46568330979	1.7207910E-22	1.5590778E-003
0.3	1.75481644985	1.75481644985	4.0558754E-22	4.8342409E-004
0.4	2.08855457670	2.08855457670	4.4666596E-22	5.1090843E-004
0.5	2.47308190605	2.47308190605	4.5211387E-22	8.2478268E-004
0.6	2.91539008062	2.91539008062	4.5171485E-22	4.9939072E-004
0.7	3.42337960270	3.42337960270	4.1874789E-22	1.1375748E-004
0.8	4.00597367129	4.00597367129	1.9556905E-22	5.4383673E-004
0.9	4.67324591120	4.67324591120	5.3402052E-22	3.0887023E-004
1.0	5.43656365692	5.43656365692	2.3064684E-21	1.0260007E-003

Example 9: Let the VIE of 1st kind (Maleknejad et al., 2011):

$$\int_0^x e^{x-t} f(t)\,dt = \sin(x), x \in [0,1] \quad (24)$$

The exact solution of this integral equation (24) is $f(x) = \cos(x) - \sin(x)$. Using Charlier polynomials as the trial function, we approximated the solution for $n = 10$, with an error of the order 10^{-16}. For this problem, Maleknejad et al. used Bernstein approximation to obtain the numerical results (Maleknejad et al., 2011). For different polynomial orders, the error in each case was

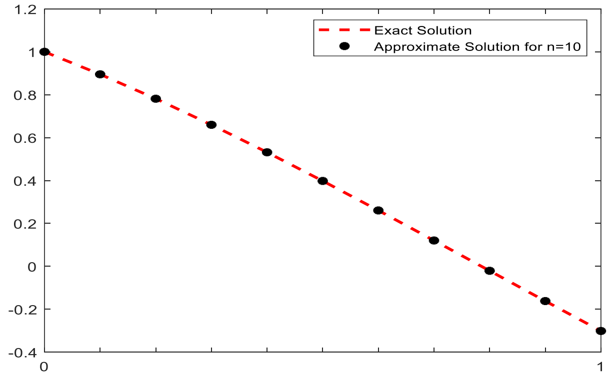

Figure 4. Exact and Approximate solutions of Example 9

determined. They found the error to be up to the order of 10^{-7} by taking $n=10$. The approximate results using Charlier polynomials are shown in Table 6, while the graphical representation of the solution is shown in Figure 4.

Table 5: Approximate Results and Comparison for Example 8

		Charlier polynomial **(Proposed)** $n=10$		Bernstein Polynomial (Zarnan & Hameed, 2024) $n=10$
x	**Exact solution**	**Approximate**	**Absolute Error**	**Absolute Error**
0.0	0.00000000000	-0.00000000000	3.1348821E-14	0.000000000000
0.1	0.09983341665	0.09983341665	8.9959762E-15	2.0508063E-012
0.2	0.19866933080	0.19866933080	7.3750428E-15	8.3558996E-013
0.3	0.29552020666	0.29552020666	7.6563870E-15	2.0756558E-013
0.4	0.38941834231	0.38941834231	5.4789858E-15	2.4960310E-013
0.5	0.47942553860	0.47942553860	8.3178149E-17	3.6565473E-013
0.6	0.56464247340	0.56464247340	5.3424722E-15	1.5317015E-013
0.7	0.64421768724	0.64421768724	7.5830547E-15	1.1908461E-013
0.8	0.71735609090	0.71735609090	7.3386471E-15	2.5211375E-013
0.9	0.78332690963	0.78332690963	8.8733859E-15	2.8431391E-013
1.0	0.84147098481	0.84147098481	3.0612577E-14	8.8095244E-013

Table 6: Approximate Results of Example 9

		Charlier polynomial **(Proposed)** $n=10$	
x	**Exact Solution**	**Approximate Solution**	**Absolute Error**
0.0	1.00000000000	1.00000000000	2.4195097E-14
0.1	0.89517074863	0.89517074863	6.1061647E-15
0.2	0.78139724705	0.78139724705	4.3205620E-15
0.3	0.65981628246	0.65981628246	6.0172041E-15
0.4	0.53164265169	0.53164265169	7.4204821E-15
0.5	0.39815702329	0.39815702329	5.6967874E-15
0.6	0.26069314151	0.26069314151	4.6806192E-16
0.7	0.12062450005	0.12062450005	6.0010285E-15
0.8	-0.02064938155	-0.02064938155	1.1771176E-14
0.9	-0.16171694136	-0.16171694136	1.5314164E-14
1.0	-0.30116867894	-0.30116867894	3.0595980E-13

Example 10: Consider the VIE of 2nd kind (Maleknejad et al., 2011):

$$f(x)=\cos(x)-e^{x}\sin(x)+\int_{0}^{x}e^{x}f(t)dt\ ;0\le x\le 1 \qquad (25)$$

The exact solution of this integral equation is $f(x)=\cos(x)$. Using Charlier polynomial as basis function, we found the approximate results with an error of the order 10^{-15} for $n=10$. On the other hand, Maleknejad et al. obtained the solution using Bernstein approximation for different polynomial orders (Maleknejad et al., 2011). In this case, the error was found to be up to the order of 10^{-10} for $n=10$. The approximate results using Charlier polynomial are shown in Table 7, and the graphical representation of the solution is shown in Figure 5.

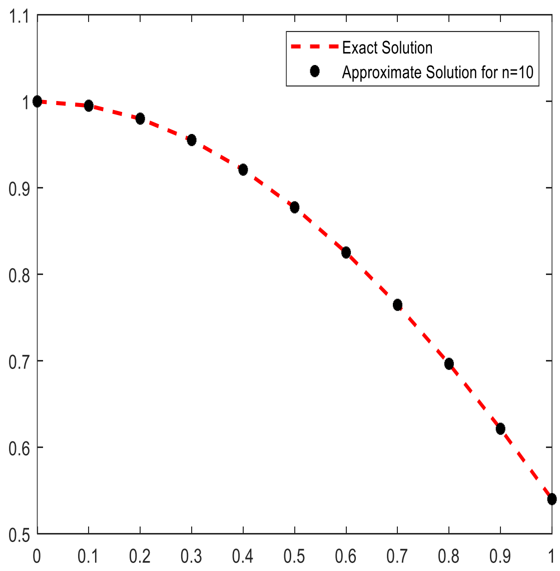

Figure 5. Comparison between approximate and exact solutions of Example 10

Table 7: Approximate Results of Example 10

		Charlier polynomial **(Proposed)** $n=10$	
x	**Exact Solution**	**Approximate Solution**	**Absolute Error**
0.0	1.00000000000	1.00000000000	1.6781873E-14
0.1	0.99500416528	0.99500416528	4.9389636E-15
0.2	0.98006657784	0.98006657784	4.1344782E-15
0.3	0.95533648913	0.95533648913	4.1772233E-15
0.4	0.92106099400	0.92106099400	2.7770717E-15
0.5	0.87758256189	0.87758256189	3.0765389E-16
0.6	0.82533561491	0.82533561491	3.1912415E-15
0.7	0.76484218728	0.76484218728	4.1910586E-15
0.8	0.69670670935	0.69670670935	3.8925604E-15
0.9	0.62160996827	0.62160996827	4.7799171E-15
1.0	0.54030230587	0.54030230587	1.6842011E-14

Example 11: Consider the VIE of 2nd kind (Berenguer et al., 2009):

$$x(t)=1+\int_0^t (s-t)x(s)ds \qquad t\in[0,1] \tag{26}$$

The exact solution of this second-kind equation is $x(t)=\cos(t)$. Using Charlier polynomials as trial function, we found the approximate results with an error of the order 10^{-77} for $n=40$. On the other hand, Berenguer et al. approximated the solution for $n=65,\ m=2$ using Schauder bases and the Geometric Series theorem (Berenguer et al., 2009). The results and comparison between Charlier polynomials and Schauder bases and Geometric Series theorem are shown in Table 8. Additionally, the graphical representation of the solution is shown in Figure 6.

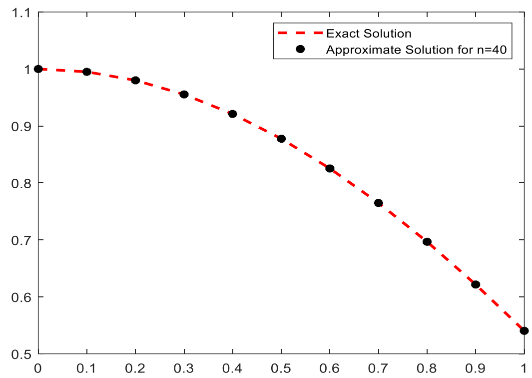

Figure 6. Comparison between approximate and exact solutions of Example 10

Table 8: Approximate Results and Comparison with others for Example 11

		Charlier polynomial **(Proposed)** $n=40$		Geometric Series theorem (Berenguer et al., 2009) $n=65, m=2$
x	**Exact Solution**	**Approximate Solution**	**Absolute Error**	**Absolute Error**
0.0	1.00000000000	1.00000000000	3.3314823E-74	---------------
0.1	0.99500416528	0.99500416528	3.8086696E-75	9.97×10^{-8}
0.2	0.98006657784	0.98006657784	4.6341777E-75	4.16×10^{-7}
0.3	0.95533648913	0.95533648913	4.2620280E-75	8.77×10^{-7}
0.4	0.92106099400	0.92106099400	3.6636261E-75	1.58×10^{-6}
0.5	0.87758256189	0.87758256189	4.5469368E-77	6.76×10^{-7}
0.6	0.82533561491	0.82533561491	3.7231146E-75	3.35×10^{-6}
0.7	0.76484218728	0.76484218728	4.2820137E-75	4.48×10^{-6}
0.8	0.69670670935	0.69670670935	4.6954986E-75	5.52×10^{-6}
0.9	0.62160996827	0.62160996827	3.8255207E-75	6.80×10^{-6}
1.0	0.54030230587	0.54030230587	3.4057771E-74	2.61×10^{-4}

Conclusion

In this paper, we have employed the Galerkin weighted residual method with Charlier polynomials as basis functions to approximate solutions of Volterra integral equations of both first- and second-kind with regular and singular kernels. A simple and rigorous matrix formulation is carried out to get approximate solutions via the Galerkin method with Charlier polynomial bases. Actually, we have concentrated our attention not only on the performance of the results but also on the matrix formulation. Several numerical examples are tested to verify the effectiveness and reliability of the proposed method. The numerical results of our proposed method demonstrate that our proposed method yields results that coincide with the exact solutions in many cases, and in other cases, the results are in good agreement with the exact solutions. In addition, the obtained numerical results confirm that our method achieves greater accuracy than the results obtained by state-of-the-art methods. Thus, we can conclude that this method may be applied to solve other types of integral equations as well as systems of integral equations.

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Conflict of Interest

The authors confirm that there is no conflict of interest with the publication of this article.

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