Volume 22, Issue 1: KUS: 1320: 68-76: 21112024: 2025

KHULNA UNIVERSITY
STUDIES

journal homepage: www.kus.ku.ac.bd

Research article

Comparative Study of Game Theory and Linear Programming in the context of a Transportation Problem

Anika Rahman Shapnil, Noboneeto Noholee, Rakibul Alam, Md. Abid Hasan, Nishat Khan and Md. Shapan Miah*

Department of Mathematics, University of Dhaka, Dhaka-1000, Bangladesh

ABSTRACT

Game theory is a modern branch of mathematics that provides a powerful framework for strategizing and analyzing situations previously which is difficult to represent mathematically. In this paper we investigate the decision-making strategies of two competing bus companies, employing game theory as the primary analytical framework. Additionally, we employ linear programming models to support the findings derived from game theory techniques. By integrating these methods, we aim to offer a comprehensive analysis of competitive strategies and optimal decision-making for both companies, thereby demonstrating the practical applications of mathematical theories in real-world scenarios. The main purpose of this paper is to explore the efficiency of two methods named Game theory and Linear programing on the contrast of a transportation problem.

ARTICLE INFO

Article timeline:
Date of Submission:
21 November, 2024
Date of Acceptance:
23 June, 2025
Article available online:
30 June, 2025

***Keywords*:**
Decision-making; game theory; linear programming; Payoff; Matrix.

Introduction

In competitive transportation systems, effective scheduling is essential for maximizing service efficiency and passenger satisfaction. Mathematical tools like game theory and linear programming provide structured methods for analyzing such decision-making problems. Game theory models strategic interactions between competitors, while linear programming focuses on optimizing outcomes under defined constraints.

Despite their widespread use, few studies directly compare these approaches within the same problem setting. This paper addresses that gap by examining a scheduling problem involving two competing bus companies, both aiming to maximize daily passenger numbers under specific operational constraints

Literature Review

Game theory has increasingly been utilized to model complex interactions in transportation systems. For instance, Shen et al. (2024) proposed a game-theoretic lane-changing decision model for autonomous vehicles, enhancing safety and efficiency in traffic flow. Similarly, Zhang et al. (2023) developed a decision-making framework that accounts for drivers' social value orientations, improving autonomous vehicles' adaptability in mixed traffic environments. Moreover, Zambrano et al. (2025) introduced a user-friendly game-theoretic modeling tool for multi-modal transportation systems, facilitating stakeholder analysis and policy-making.

While both Game Theory and Linear Programming have been applied extensively in transportation research, there is a paucity of studies directly comparing these methodologies within the same decision-making context. Most existing research focuses on either strategic interactions (Game Theory) or optimization problems (Linear Programming) in isolation. This approach limits the understanding of how these methods perform relative to each other in practical scenarios, such as transportation scheduling problems involving multiple stakeholders with competing objectives.

The current study addresses this gap by conducting a comparative analysis of Game Theory and Linear Programming in the context of a transportation scheduling problem. By applying both methodologies to the same problem, the study provides insights into their respective strengths and limitations, offering a more holistic understanding of decision-making tools in transportation planning. This comparative approach is particularly valuable for policymakers and practitioners seeking to select appropriate analytical methods for complex transportation issues.

*Corresponding author: <shapan@du.ac.bd>

DOI: https://doi.org/10.53808/KUS.2025.22.01.1320-se

Required Theorems

Theorem 1

In a mixed-strategy Nash equilibrium [10], each player must be indifferent among all the pure strategies that they are mixing. That is, each of these pure strategies must yield the same expected payoff, given the strategies of the other players (Indifference Theorem).

For a matrix game with $m \times m$ matrix A, if Player 1 uses the mixed strategy $p = (p_1, \ldots, p_m)^T$ and Player 2 uses column j, Player 1's average payoff is $\sum_{i=1}^{m} p_i a_{ij}$. If V is the value of the game, an optimal strategy, $p,$ for 1 is characterized by the property that Player 1's average payoff is at least V no matter what column j Player 2 uses, that is

$$\sum_{i=1}^{m} p_i a_{ij} \geq V \quad \forall\, j = 1, \ldots, n$$

Theorem 2

Any two dual games [11] are always feasible with a bounded feasible region and thus have optimal solutions with a common optimal value, namely the value of the game. In other words, for every finite two-player zero-sum game, the primal game and its dual game have the same value (Duality in Game Theory).

Theorem 3

For a primal problem, $Max\ \boldsymbol{z} = \mathbf{c}^{\mathrm{T}}\mathbf{x}$ subject to $A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq 0$ and its dual problem $Min\ \boldsymbol{w} = \mathbf{b}^{\mathrm{T}}\mathbf{y}$ subject to $A^T\boldsymbol{y} \geq \boldsymbol{c}, \boldsymbol{y} \geq 0$, if x_0 and y_0 are feasible solutions to the primal and dual problems respectively, the following inequality holds,

$$\mathbf{c}^{\mathrm{T}}\mathrm{x}_0 \leq \mathbf{b}^{\mathrm{T}} y_0$$

This is known as the Weak Duality Theorem [12]. The weak duality theorem ensures that no matter what the feasible solutions x_0 and y_0 are chosen, the value of primal objective function will never exceed the value of the dual objective function.

Theorem 4

If there are feasible solution x^* and y^* such that

$$\mathbf{c}^{\mathrm{T}}\mathrm{x}^* = \mathbf{b}^{\mathrm{T}}\mathrm{y}^*$$

This is known as the Strong Duality Theorem [13,14]. $x*$ and $y*$ are optimal solutions to the primal and dual problems respectively. Strong duality theorem tells us that, when both problems have optimal solutions, the maximum value of the primal objective function will equal the minimum value of the dual objective function, which provides a direct relationship between the two problems.

Theorem 5

If a dual variable is greater than zero or slack then the corresponding primal constraint must be an equality and if the primal constraint is slack then the corresponding dual variable is tight. This is known as Complementary Slackness Theorem [15].

A Transportation Problem

Two private inter-city bus companies, let's call them Company A and Company B, are about to operate their services. The companies are required to meet some criteria by the license-providing department of the authorities.

Let the specifications be as follows

- Both companies must schedule 5 buses per day
- The time interval between the departure of two buses must be 3 hours for both companies
- The departures are at the beginning of an hour

For the sake of simplicity, we assume some more criteria for the passengers, which are mentioned below

- 20 passengers gather at the bus station at each hour
- The passengers split equally between two companies buses if both buses depart at the same time
- If a passenger has to wait for more than 3 hours, s/he will take a different mode of transportation
- The passengers will take the next earliest bus if there are no immediate buses to get on

The only decision left for companies A and B to make is when they should start in the morning. They want to maximize the number of passengers on their buses as each company's payoff is proportional to that number.

Let the possible schedules be –

S1: the first bus departs at 06:00, so the later ones depart at 09:00, 12:00, 15:00, and 18:00 respectively

S2: the first bus departs at 07:00, so the later ones depart at 10:00, 13:00, 16:00, and 19:00 respectively

S3: the first bus departs at 08:00, so the later ones depart at 11:00, 14:00, 17:00, and 20:00 respectively

S4: the first bus departs at 09:00, so the later ones depart at 12:00, 15:00, 18:00, and 21:00 respectively

We now demonstrate how to calculate the number of passengers boarding each bus.

Assume that Company A follows S1, that is, its buses leave at 06:00, 09:00, 12:00, 15:00, and 18:00. Let Company B's buses leave at 07:00, 10:00, 13:00, 16:00, and 19:00; thus, they follow S2.

The following table shows the most convenient buses for the passengers at each hour, keeping in mind that no one will be waiting for a very long time. The last row of the table shows the total number of passengers on each bus leaving the station at the corresponding times.

Table 1: Passengers on each bus of a day if Company A follows the S1 schedule and Company B follows the S2 schedule

Time	Bus Preference	No. of Passengers
6:00	A	20
7:00	B	20
8:00	A	-
9:00	A	40
10:00	B	20
11:00	A	-
12:00	A	40
13:00	B	20
14:00	A	-
15:00	A	40
16:00	B	20
17:00	A	-
18:00	A	40
19:00	B	20
20:00	-	-
21:00	-	-

Evidently, Company A will transport a total of 180 passengers daily whereas Company B will only be able to transport 100 passengers.

Now, if we analyze all the possible pairs of schedules for Company A and Company B, then we get the following normal form.

Table 2: Payoff Bimatrix

A / B	S1	S2	S3	S4
S1	130, 130	180, 100	100,180	140, 180
S2	100, 180	140, 140	200, 100	120, 200
S3	180, 100	100, 200	150, 150	220, 100
S4	180, 140	200, 120	100, 220	160, 160

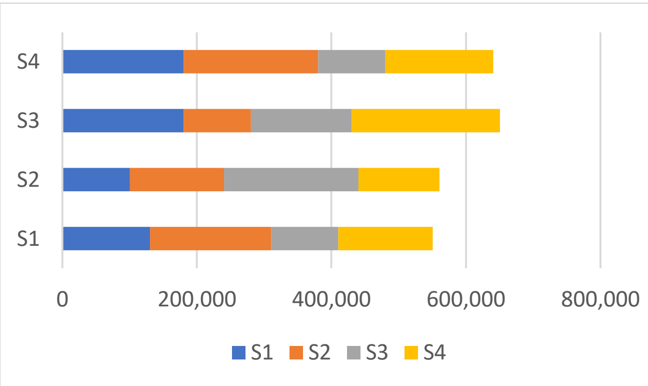

Figure 1: Graphical Representation of Bi-matrix Payoff of two companies.

Solution by Game Theory

The oddment method, also known as the method of oddments, is a technique used in game theory to identify optimal strategies in a two-person zero-sum game with no saddle point. By transforming payoff matrices and calculating oddments, players can identify optimal strategies that ensure balanced outcomes. Despite its limitations in scalability and sensitivity, it remains a fundamental method for understanding strategic interactions in competitive environments.

Oddment Method for n × n Games

Consider M = (a_{ij}) as an n × n payoff matrix.
For n = 3,

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Step 1:
  Generate one new matrix *C* such that its first column is formed by subtracting the second column from the first column of the matrix *M*. The second column of new matrix *C* is then obtained by subtracting the third column from the second, and this process continues until the last column of *M* is processed. Consequently, *C* becomes an $n \times (n-1)$ matrix.
  $$C = \begin{bmatrix} a_{11} - a_{12} & a_{12} - a_{13} \\ a_{21} - a_{22} & a_{22} - a_{23} \\ a_{31} - a_{32} & a_{32} - a_{33} \end{bmatrix}$$
- Step 2:
  From matrix *M*, construct another new matrix *R* by subtracting each subsequent row from the previous row, analogous to the column subtraction process performed in Step 1. This results in *R* being an $(n-1) \times n$ matrix.
  $$R = \begin{bmatrix} a_{11} - a_{21} & a_{12} - a_{22} & a_{13} - a_{23} \\ a_{21} - a_{31} & a_{22} - a_{32} & a_{23} - a_{33} \end{bmatrix}$$
- Step 3:
  Calculate the oddments for each row and column of the matrix *M*. The oddment for the $i-th$ row is the determinant of C_i, where C_i is the matrix *C* with the $i-th$ row removed. In the same way, the oddment for the $j-th$ column is the determinant of R_j, where R_j is the matrix R with the $j-th$ column removed.
  $$C_1 = \begin{bmatrix} a_{21} - a_{22} & a_{22} - a_{23} \\ a_{31} - a_{32} & a_{32} - a_{33} \end{bmatrix}$$
  $$R_2 = \begin{bmatrix} a_{11} - a_{21} & a_{13} - a_{23} \\ a_{21} - a_{31} & a_{23} - a_{33} \end{bmatrix}$$
- Step 4:
  Write the absolute values of these oddments next to their corresponding rows and columns.
- Step 5:
  Verify if
  $$\sum \text{Row Oddments} = \sum \text{Column Oddments}.$$
  If the sum of the row oddments is equal to the sum of the column oddments, the oddments can be expressed as fractions of the total, revealing the optimal strategies.
  Otherwise, the method is considered unsuccessful.
- Step 6:
  Determine the game's expected value using the optimal mixed strategy for the row player,

applicable against any strategy of the column player.

To use the Oddment method, we first transform the payoff bimatrix (Table 2) into a zero-sum form. To do this, we take the differences of the two values in each cell, considering Company A as the primary.

Table 3: Zero-Sum Game Representation for both companies

Company A / Company B	S1	S2	S3	S4
S1	0	80	-80	-40
S2	-80	0	100	-80
S3	80	-100	0	120
S4	40	80	-120	0

Probabilities of the pure strategies in the mixed strategy, Mix-1 are as follows

$$P(S1) = \frac{240000}{544000} \approx 44\%$$

$$P(S2) = \frac{64000}{544000} \approx 12\%$$

$$P(S3) = \frac{32000}{544000} \approx 6\%$$

$$P(S4) = \frac{208000}{544000} \approx 38\%$$

We now calculate the expected payoff of Company A for Mix-1.

For Company A,

If S1 is chosen then the expected payoff is,

$$(0.44 \times 130) + (0.12 \times 180) + (0.06 \times 100) + (0.38 \times 140) = 138$$

If S2 is chosen then the expected payoff is,

$$(0.44 \times 100) + (0.12 \times 140) + (0.06 \times 200) + (0.38 \times 120) = 118.4$$

If S3 is chosen then the expected payoff is,

$$(0.44 \times 180) + (0.12 \times 100) + (0.06 \times 150) + (0.38 \times 220) = 183.8$$

If S4 is chosen then the expected payoff is,

$$(0.44 \times 180) + (0.12 \times 200) + (0.06 \times 100) + (0.38 \times 160) = 170$$

Interpretation of the results are

- The expected payoff is 138 for choosing strategy S1. This means that if Company A chooses strategy S1, it can expect to transport an average of 138 passengers daily
- Choosing strategy S2, the company can expect to receive 118 passengers daily
- Choosing strategy S3, the expected daily payoff is 183.8 which means the bus can carry an average of 183 passengers daily
- Choosing strategy S4, it can transport an average of 170 number of passengers daily

For Company B we will get the same results as Company A.

Based on the mixed strategy Mix-1, Companies A & B can maximize their payoffs by choosing strategy S3, which yields an expected payoff of 183.8.

Solution by Linear Programming

The Simplex Algorithm

The simplex algorithm is a method in linear programming that finds the optimal solution to a system of linear equations with constraints, by focusing on a selected few solutions instead of checking all possible ones. It starts with an initial feasible solution and moves to adjacent feasible solutions step-by-step, improving the objective function at each stage. In each iteration, the algorithm selects a variable to enter the solution (pivot) and adjusts the solution to ensure both feasibility and improvement in the objective function. This process repeats until no further improvements can be made, meaning the optimal solution has been reached. Notably, all solutions are located at the corner points of the feasible region.

Here's a mathematical explanation of the simplex algorithm:

(i) Objective Function and Constraints:

In a linear programming problem, the goal is to maximize (or minimize) an objective function, subject to certain constraints. The objective function can be written as:

$$\max Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

where $(c_1, c_2, \ldots, c_n)$ are the coefficients of the variables $x_1, x_2, \ldots, x_n$ respectively in the objective function, and Z is the value of the objective function.

The constraints are represented as linear equations or inequalities:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$

$$\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$$

Where,

a_{ij} ($i = \overline{1, m}$ $j = \overline{1, n}$) are coefficients of the variables in the constraints,

b_i ($i = \overline{1, m}$) are constants on the right-hand side, and

m is the number of constraints.

(ii) Initialization:

The algorithm begins with an initial feasible solution, usually obtained by setting some variables to zero and solving for the other variables.

(iii) Entering and Leaving Variables:

At each iteration, one variable enters the solution (entering variable), and another exits (leaving variable). These variables are chosen based on the objective function and the constraints.

(iv) Pivoting:

After selecting the entering and leaving variables, the solution is adjusted by pivoting to move to an adjacent feasible solution, improving the objective function while maintaining feasibility.

(v) Termination:

The method keeps iterating through the addition and subtraction of variables, turning at each turn, until the objective function can no longer be improved. The algorithm ends here since the current result is the best one.

(vi) Optimality and Feasibility:

The algorithm makes sure that the solution stays feasible (that is, it meets all the constraints) and optimal (that is, it either maximizes or minimizes the objective function).

To sum up, the simplex algorithm improves the objective function by methodically examining the corners of the viable region until an ideal solution is discovered.

Dual Problem

In the context of linear programming, the dual problem is associated with a given primal problem which involves the problem of maximizing or minimizing an objective function with a different set of linear constraints. Here is a mathematical summary of the dual problem.

The primal linear programming problem is presented in its standard form as:

$$\text{Max } \mathbf{Z} = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

Subject to:

$$\mathrm{A}\mathbf{x} \le \mathbf{b}$$

$$\mathbf{x} \ge 0$$

Where,

- Z is the objective function to be maximized.
- $\boldsymbol{c}$ is the coefficient vector in the objective function.
- $\boldsymbol{x}$ represents the vector of decision variables.
- A is the matrix of coefficients for the constraints.
- $\boldsymbol{b}$ is the vector of constants in the constraints.
- $\boldsymbol{x} \ge 0$ indicates that the decision variables are non-negative.

The dual problem is associated with the primal problem which involves minimizing a new objective function with a different set of constraints. It is formulated as follows:

$$\text{Min } \mathbf{W} = \mathbf{b}^{\mathrm{T}}\mathbf{y}$$

Subject to:

$$\mathrm{A}^{\mathrm{T}}\mathbf{y} \ge \mathbf{c}$$

$$\mathbf{y} \ge 0$$

Where:

- W represents the objective function of the dual problem.
- $\boldsymbol{y}$ indicates the vector of dual variables.
- A^T represents the transpose of matrix A
- $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors.
- $\boldsymbol{y} \ge 0$ indicates that the dual variables are non-negative.

Formulating the mixed strategy Mix-2 we get,

Let V be the value of the game, whose value will be bounded between the maximin (-80) and minimax (80), that is,

Maximin value ≤ Game value ≤ Minimax value.

Since p_i and q_i are probabilities, they are all non-negative. Also from the laws of probability, we have $\sum_{i=1}^{4} p_i = 1$ and $\sum_{i=1}^{4} q_i = 1$.

Now, when Company B plays strategy S1, A will have the game value of $-80p_2 + 80p_3 + 40p_4$ which must be at most equal to the game value V. Therefore, $-80p_2 + 80p_3 + 40p_4 \ge V \Rightarrow V + 80p_2 - 80p_3 - 40p_4 \le 0$, which acts as a constraint.Similarly, when Company B plays strategies S2, S3, and S4, we get the other three constraints for Company A.

The linear programming problem (LPP) equivalent to our problem is then,

$$\text{Maximize } Z = V$$

Subject to,

$$p_1 + p_2 + p_3 + p_4 = 1$$

$$V - 0p_1 + 80p_2 - 80p_3 - 40p_4 \leq 0$$

$$V - 80p_1 - 0p_2 + 100p_3 - 80p_4 \leq 0$$

$$V + 80p_1 - 100p_2 - 0p_3 + 120p_4 \leq 0$$

$$V + 40p_1 + 80p_2 - 120p_3 - 0p_4 \leq 0$$

$$p_i \geq 0 \ (i = 1,2,3,4) \text{ and V unrestricted in sign}$$

We solved this linear program in AMPL and got the following results.

$V = 0 \quad p_1 = 0 \quad p_2 = 0.4 \quad p_3 = 0.266667 \quad p_4 = 0.33333$

Now, the equivalent LPP for Company B is the dual of the LPP for Company A.

$$\text{Minimize } W = V$$

Subject to,

$$q_1 + q_2 + q_3 + q_4 = 1$$

$$V + 0q_1 - 80q_2 + 80q_3 + 40q_4 \geq 0$$

$$V + 80q_1 + 0q_2 - 100q_3 + 80q_4 \geq 0$$

$$V - 80q_1 + 100q_2 + 0q_3 - 120q_4 \geq 0$$

$$V - 40q_1 - 80q_2 + 120q_3 + 0q_4 \geq 0$$

$$q_i \geq 0 \ (i = 1,2,3,4) \text{ and V unrestricted in sign}$$

Which would give the same objective functional value.

Therefore, the mixed strategy Mix-2 thus found is,

$0\% \quad of \quad S1$

$40\% \quad of \quad S2$

$26.67\% \quad of \quad S3$

$33.33\% \quad of \quad S4$

For Company A:

If S1 is chosen then the expected payoff is,

$$(0 \times 130) + (0.40 \times 180) + (0.2667 \times 100) + (0.3333 \times 140) = 145.332 \approx 145$$

If S2 is chosen then the expected payoff is,

$$(0 \times 100) + (0.40 \times 140) + (0.2667 \times 200) + (0.3333 \times 120) = 149.336 \approx 149$$

If S3 is chosen then the expected payoff is,

$$(0 \times 180) + (0.40 \times 100) + (0.2667 \times 150) + (0.3333 \times 220) = 153.331 \approx 153$$

If S4 is chosen then the expected payoff is,

$$(0 \times 180) + (0.40 \times 200) + (0.2667 \times 100) + (0.3333 \times 160) = 159.998 \approx 160$$

For Company B we will get the same result.

Based on the mixed strategy Mix-1, Company A can maximize its expected payoff by choosing strategy S4, which yields an expected payoff of 159.998. Company B can maximize its expected payoff by choosing strategy S4, which yields an expected payoff of 160.002 under Mix-2.

Results and Comparative Analysis

Now we consider all the strategies at once and find out the best strategies for companies A and B:

Table 4: Payoff Matrix with All the Pure and Mixed Strategies

B \ A	S1	S2	S3	S4	Mix-1	Mix-2
S1	**130, 130**	180, 100	100, 180	140, 180	138, 148.4	145.3, 148
S2	100, 180	**140, 140**	200, 100	120, 200	118.4, 178	149.3, 149.3
S3	180, 100	100, 200	**150, 150**	220, 100	183.8, 115	153.3, 153.3
S4	180, 140	200, 120	100, 220	**160, 160**	170, 150	160, 160
Mix-1	148.4, 138	178, 118.4	115, 183.8	150, 170	**150.56, 150.56**	150.56, 153.93
Mix-2	148, 145.3	149.3, 149.3	153.3, 153.3	160, 160	153.96, 150.56	**153.93, 153.93**

Observations:

- Among the mixed strategies, we get equilibriums when both companies choose the same mixed strategy. The Nash equilibrium is Mix-2 vs Mix-2 gives the highest payoffs for both companies
- The individual best payoffs for the companies among all the strategies (excluding Mix-1) are obtained when

the companies choose S3 (payoff of 220), given the other company chooses S4.

- However, the payoff will change according to the choice of the opponent. For example, if one chooses S3 but the other company chooses any other strategy than S4, then the first company may have the least payoff of 100.

Taking a decision without knowing the other company's decision does not guarantee the best payoff for either. Hence, non-cooperative games are always a gamble. To ensure a better payoff for both companies, their best bet is to cooperate and communicate.

To ensure better payoff for both companies, their best bet is to cooperate and communicate.

While Linear Programming ultimately gave the strategy that is most beneficial for both companies, the mixed strategy calculated explicitly with concepts from Game Theory wasn't far behind. At equilibria, the difference of expected numbers of passengers is just of $153.96–150.56 = 3.4 \approx 3$. We've used the concepts of Game Theory to formulate our linear programming problem in a way that considers each scenario for both companies.

Linear programming typically optimizes resource allocations, it lacks the ability to account for the competitive dynamics of real-world problems. For a larger problem it'd be easier to implement the oddment method than the simplex method. Both fields have immense success in solving real-life decision making problems. Rather than arguing one's superiority over the other, we are excited to experience the developments of both.

This problem has also showed us that cooperation among individuals and even different fields of study might just be the best strategy of them all!

In the transportation problem we studied, two cases may arise.

Case 1: Maximizing Profit with Risk

In this scenario, both companies aim to maximize their profits. The payoff matrix indicates that the highest number of passengers each company can attract in a day is 220 by selecting the S3 schedule. However, this choice carries significant risk. For example, if Company A opts for S3, hoping to secure 220 passengers, but Company B chooses S2 instead, Company A could end up with only 100 passengers—120 less than expected. The same risk applies to Company B.Thus, while choosing the S3 schedule offers the potential for maximum profit, it also poses the risk of substantial losses.

Case 2: Minimizing Risk for Steady Payoff

In this case, both companies prefer to avoid risk and aim for a stable payoff over the long term. To achieve this, both companies would likely adopt the mixed strategy Mix-3, which ensures the highest possible payoff for both parties.

If this were a cooperative game-allowing the bus companies to communicate before making decisions-they would likely reach an agreement that maximizes their payoffs. In this scenario, both companies would probably settle on the (S4, S4) strategy, which guarantees them the highest equal payoff. As we can see in the figures, the payoffs might reach a higher value, but they fluctuate a lot in the four pure strategies. Although Mix-1 is more stable than the pure strategies, Mix-2 gives the most evenly distributed payoffs. Hence, Mix-2 seems to be the safest bet and the Mix-2 vs Mix-2 equilibrium can be considered as the Nash equilibrium.

Both company aims to get the maximum number of passengers with the least inherent risk. In this regard, the equilibria are the safest choice because in an equilibrium, every player's strategy is the best one given the other players' strategies. Figure 1 solidifies this notion because it illustrates that the payoffs are the closest at the equilibria. Table 4 shows that the equilibria are at the diagonal of the payoff bi-matrix.

At first glance, the S4 vs S4 equilibrium seems to give the highest payoff. But is it the best strategy? Suppose Company A chooses S4 in hopes of securing the higher payoff. The problem is that they don't know which strategy Company B would be playing. They could get either 200 or 100 passengers based on whether the other company plays S2 or S3. A similar problem exists from Company B's perspective. Thus, the equilibria of the pure strategies are not in fact the best strategies.

Both mixed strategies have an equilibrium, and the payoffs don't vary as much as the pure ones. Between the two, Mix-2 has a lower risk of losing passengers, no matter what the other company chooses to do.

Conclusion

This study introduces a game-theoretic approach to analyzing the decision-making strategies of two competing bus companies, emphasizing both risk and profit trade-offs in non-cooperative scenarios. Unlike traditional transportation studies that often focus on operational efficiency or market competition, this research highlights the strategic interplay between payoff maximization and risk minimization.

The analysis reveals distinct decision-making strategies for the competing bus companies, highlighting the interplay between maximizing profits and minimizing risks. The highest payoffs for both companies are achieved when they adopt the same mixed strategy, specifically Mix-2 versus Mix-2. This represents a Nash equilibrium, ensuring mutual benefit within the competitive framework.

While the S3 schedule offers the potential for the maximum individual payoff of 220 passengers, this strategy carries significant risk. If the opposing company selects a different schedule, such as S2, the payoff can drop dramatically, demonstrating the trade-off between high reward and high risk.

For companies prioritizing stability over potential profit spikes, adopting the Mix-3 strategy provides the most consistent outcomes. This approach mitigates the risk of significant losses and ensures a reliable payoff for both parties over time.

If the game allowed for collaboration between the companies, they would likely converge on a mutually beneficial strategy, such as (S4, S4). This cooperative approach would guarantee equal and optimal payoffs for both companies.

Overall, the choice of strategy is contingent on each company's priorities-whether they seek to maximize profits at the expense of potential risk or prefer a steady and predictable outcome. The analysis underscores the complexity of decision-making in competitive environments and the value of game theory in identifying optimal strategies.

The dual focus on mixed strategies and scenario-based outcomes using game theory offers fresh perspectives on competitive strategy formulation in the transportation sector, paving the way for further applications of game theory in similar multi-agent decision environments. There are many limitations in our work like limited data collection and choice of methods. In future we will work to eliminate these kinds of limitations.

Acknowledgement

We are very much grateful to our supervisor Md. Shapan Miah and to department of mathematics for providing necessary supports.

Competing Interest

The authors report that there are no competing interests to declare.

References

Alexander V. Kuznetsov, Andrew Schumann & Małgorzata Rataj. (2024). Continuous optimisation problem and game theory for multi-agent pathfinding,Volume 53, pages 1–41

André Casajus & Rodrigue Tido Takeng. (2024). Second-order productivity, second-order payoffs and the Banzhaf value, Volume 53, Pages: 989 – 1004

Bertsimas, D. , Tsitsiklis, J. N. (1997). Introduction to Linear Optimization,

Frederick, S. H. & Gerald, J. L. (2020). Introduction to Operations Research, McGraw-Hill,

George B. Dantzig. (1947). Linear Programming and Extensions.

George, B. D. (1951). Applications of Linear Programming in Optimization, *Operations Research Journal,*

George, H., A. (1980). Note on the Applicability of the Method of Oddments to 3X3 Two Person Zero Sum Games, *The Journal of the Operational Research Society*: Vol 31, No. 4.

Hiller, F. S. , Liberman, G. J. , (2014). Introduction to Operation Research, Mc-Graw Hill,

Karlin, S. , (1959). Mathematical Methods and Theory in Games, Programming, and Economics,

Luce, R. D. Raiffa, H. , (1957).Games and Decisions : Introduction and Critical Survey,

Mansour, A. , Atefeh, A. , Reza, M. ,& Javid, J. , A . (2022). Game Theory Approach to Selecting Marketing-Mix Strategies, *Journal of Advances in Management Research* : Vol. 19 No. 1.

Marc-Olivier, B. , Jean, L.& , Didier, B.(2023). Game Theory: An Appropriate Tool for Strategic Analysis?, *Revue Défense Nationale* : 858

Martin J. Osborne, & Ariel Rubinstein. (1994). A Course in Game Theory, MIT Press

Pablo Amorós, (2024). Nash implementation of supermajority rules, Published: 23 February 2024 ,Volume 53, pages 811–825,

Roger B. Myerson. (1991). Game Theory: Analysis of Conflict, Harvard University Press.

Steven Tadelis, (2013). Game Theory: An Introduction, Princeton University Press,

Taha H. A. (2016). Operations Research: An Introduction, Pearson,

Vanderbei, R. J. , (2020).Linear Programming: Foundations and Extensions, *Springer*.

Nomenclature:

GT	game theory
LPP	linear programming problem
AMPL	a mathematical programming language
A,B	two bus companies for competitive scenario
V	value of the game
R	row matrix
C	column matrix
OM	oddment method
MS	mixed strategies
DP	dual problem
PP	primal problem
Vmin	maximin value
Vmax	minimax value
OS	optimal solution
FS	feasible solution
NE	Nash equilibrium
S1,S2,S3,S4	strategies
z	objective function
P	payoff matrix
p=(p1,p2,p3,p4)	Mixed strategy for company A
q=(q1,q2,q3,q4)	Mixed strategy for company B
E(A)	Expected payoff for company A
E(B)	Expected payoff for company B