



A NOTE ON CONTACT MANIFOLDS AND APPLICATIONS

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Abstract: The objective of this paper is to define contact manifold in a popular way and to show its applications to non-linear system and different branch of physics. A well known class of contact manifold viz., Sasakian manifold has been studied and its applications have also been considered. An illustrative example is also given.

Keywords: Contact manifold, Sasakian manifold, conharmonic curvature tensor

Introduction

In modern engineering analysis, the methods of contact geometry play an important role. Contact geometry has evolved from the mathematical formalism of classical mechanics. The roots of contact geometry appear in the works of Christiaan Huygens, Barrow and Isaac Newton. The theory of contact transformations was developed by Sophus Lie, with the dual aims of studying differential equations and describing the 'change of space element', familiar from projective duality. An important class of contact manifolds is formed by Sasakian manifolds (Sasaki, 1960). The main objective of this paper is to study such manifolds and to observe their applications in different branch of applied mathematics. This paper is organized in the following order contact manifolds, application of contact manifolds, Sasakian manifold, conharmonically flat Sasakian manifold, study of conharmonically ϕ -symmetric Sasakian manifolds, conharmonically semi-symmetric Sasakian manifolds and discussion of applications of Sasakian manifolds. The last one gives an illustrative example of conharmonically ϕ -symmetric Sasakian manifold of dimension three.

Materials and Methods

Contact manifolds: Let M be a $(2n + 1)$ -dimensional differentiable manifold and ϕ, ξ, η be a tensor field of type $(1, 1)$, a vector field, a 1-form on M respectively. If ϕ, ξ, η satisfy the conditions

$$\eta(\xi) = 1, \quad \phi^2(X) = -X + \eta(X)\xi, \quad (1)$$

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for any vector field X on M , then M is said to have an almost contact structure (ϕ, ξ, η) and is called an almost contact manifold. For an almost contact structure (ϕ, ξ, η) , we also have

$$\phi \xi = 0, \quad \eta(\phi X) = 0, \quad \text{rank } \phi = 2n \quad (2)$$

Every almost contact manifold M admits a Riemannian metric tensor field g such that

$$\eta(X) = g(X, \xi) \quad (3)$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (4)$$

A $(2n + 1)$ – dimensional almost contact manifold is called a contact manifold if it carries a global 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0 \quad (5)$$

every where on M , where the exponent denotes n^{th} exterior power. We call η a contact form of M (Blair, 1976).

Example of contact manifold: Let us consider $R^3(x, y, z)$ with the 1-form $dz - ydx$. Let the contact plane ξ at the point (x, y, z) be spanned by the vectors

$$X_1 = \frac{\partial}{\partial y},$$

$$\text{and } X_2 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}.$$

Then R^3 becomes a contact manifold.

Applications of contact manifolds: Contact geometry has broad applications in physics, e.g., geometrical optics, classical mechanics, thermodynamics, geometric quantization, and applied mathematics, such as control theory. In the present section, we shall illustrate about some applications of contact manifolds in partial differential equation. Now, let us define (Edelen and Hua Wang, 1992), (Guojing and Jianguo, 1998), (Morimoto, 1997):

Monge-Ampere exterior differential systems: Let us first recall the notation of an exterior differential system. Let M be a differentiable manifold and let A denote the sheaf of germs of differential forms on M . An exterior differential system on M is a sub sheaf Σ of A such that

- (1) Each stalk $\Sigma_x, x \in M$, is an ideal A_x ,
- (2) Σ is closed under exterior differentiation, that is, $d\Sigma \subset \Sigma$,
- (3) Σ is locally finitely generated. (Morimoto, 1997)

Definition: An exterior differential system Σ on a contact manifold is called a Monge-Ampere exterior differential system (or simply M-A system) if Σ is locally generated by a contact form w of D and an n -form θ (Morimoto, 1997).

By a solution of a M-A system Σ we mean an integral manifold of Σ of dimension n . Note that an integral manifold of Σ is a fortiori of an integral manifold of D , namely an isotropic submanifold, and a Legendre submanifold if the dimension takes the maximum value n . Hence a solution of an M-A system is, in particular, a Legendre submanifold.

To justify the terminology, let us see that a solution of a M-A system turns out to be a solution of a so-called Monge-Ampere equation when expressed in terms of a suitable canonical coordinate system.

Let Σ be a M-A system on a contact manifold M of dimension $(2n + 1)$ and let $l : S \rightarrow M$ be a Legendre submanifold. Take a point $a \in S$. By Darboux's theorem there is a local coordinate system (called canonical coordinate system) $x^1, x^2, \dots, x^n, z, p_1, \dots, p_n$ of M around $l(a)$ such that the contact structure is locally defined by the 1-form $w = dz - \sum_{i=1}^n p_i dx^i$.

Moreover, we can choose a canonical coordinate system such that $l^* dx^1, l^* dx^2, \dots, l^* dx^n$ are linearly independent at a . Then the image $l(v)$ may be expressed in a neighbourhood v of a as a graph:

$$\begin{cases} z_1 = \phi(x^1, x^2, \dots, x^n) \\ p_j = \psi_j(x^1, x^2, \dots, x^n), \end{cases}$$

since S is a Legendre submanifold, we have

$$\psi_j = \frac{\partial \phi}{\partial x^j}, \quad j = 1, 2, \dots, n.$$

Let θ be the n -form which, together with w , generates the M-A system Σ . Write down it in the canonical coordinates as

$$\theta = \sum_{\substack{i_1 \leq \dots \leq i_n \\ j_1 \leq \dots \leq j_{n-1}}} F_{i_1 \dots i_n}^{j_1 \dots j_{n-1}} dx^{i_1} \wedge \dots \wedge dx^{i_n} \wedge dp_{j_1} \wedge \dots \wedge dp_{j_{n-1}} \pmod{w} \quad (6)$$

Then $l|_V : V \rightarrow M$ is a solution of Σ if and only if

$$\sum F_{i_1 \dots i_n}^{j_1 \dots j_{n-1}} \left(x^1, \dots, x^n, \phi, \frac{\partial \phi}{\partial x^1}, \dots, \frac{\partial \phi}{\partial x^n} \right) \Delta_{j_1 \dots j_{n-1}}^{i_1 \dots i_n}(\phi) = 0 \quad (7)$$

where $\Delta_{j_1 \dots j_{n-1}}^{i_1 \dots i_n}(\phi)$ denotes the minor of the Hessian matrix of ϕ given by

$$\Delta_{j_1 \dots j_{n-1}}^{i_1 \dots i_n}(\phi) = \text{sgn} \begin{pmatrix} 1, 2, \dots, l, l+1, \dots, n \\ i_1, i_2, \dots, i_l, k_1, \dots, k_{n-1} \end{pmatrix} \begin{vmatrix} \frac{\partial^2 \phi}{\partial x^{j_1} \partial x^{k_1}} & \dots & \frac{\partial^2 \phi}{\partial x^{j_1} \partial x^{k_{n-1}}} \\ \dots & \dots & \dots \\ \frac{\partial^2 \phi}{\partial x^{j_{n-1}} \partial x^{k_1}} & \dots & \frac{\partial^2 \phi}{\partial x^{j_{n-1}} \partial x^{k_{n-1}}} \end{vmatrix}$$

with $\{1, 2, \dots, n\} = \{l_1, \dots, i_l, k_1, \dots, k_{n-l}\}$ and $k_1 < \dots < k_{n-l}$.

A second order non-linear partial differential equation for one unknown function ϕ with n independent variables of the form (7) is known as Monge-Ampere equation. In particular, when $n = 2$, it has the following form familiar in the classical literature (Morimoto, 1997).

$$Hr + 2Ks + Lt + M + N(rt - s^2) = 0,$$

where $p = \frac{\partial \phi}{\partial x}$, $q = \frac{\partial \phi}{\partial y}$, $r = \frac{\partial^2 \phi}{\partial x^2}$,

$$s = \frac{\partial^2 \phi}{\partial x \partial y}, \quad t = \frac{\partial^2 \phi}{\partial y^2} \quad \text{and } H, K, \dots, N \text{ are function of } x, y, z, p, q .$$

Thus a Monge-Ampere equation may be considered as a coordinate representation of a more intrinsic object of a Monge-Ampere exterior differential system.

Example (Morimoto, 1997), Consider $\mathbb{R}^5(x, y, z, p, q)$ as a contact manifold equipped with a contact form

$$w = dz - pdx - qdy .$$

Let Σ be a M-A system generated by the following 2-form (and w):

$$\theta = dp \wedge dq .$$

If a solution of Σ is represented in the form

$$z = \phi(x, y), \quad p = \psi_1(x, y), \quad q = \psi_2(x, y),$$

then the function ϕ is a solution of the Monge-Ampere equation

$$rt - s^2 = 0 .$$

If we introduce new co-ordinates $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q})$ defined by

$$\bar{x} = \bar{p}, \quad p = -\bar{x},$$

$$y = \bar{y}, \quad q = \bar{q},$$

$$z = \bar{z} - \bar{p}\bar{x},$$

then we have $w = d\bar{z} - \bar{p}d\bar{x} - \bar{q}d\bar{y}$

$$\text{and } \theta = -d\bar{x} \wedge d\bar{q} .$$

Hence if a solution of Σ is represented in the form

$$\bar{z} = \bar{\phi}(\bar{x}, \bar{y}), \quad \bar{p} = \bar{\psi}_1(\bar{x}, \bar{y}), \quad \bar{q} = \bar{\psi}_2(\bar{x}, \bar{y}),$$

the function $\bar{\phi}$ satisfies Monge-Ampere equation

$$\bar{t} = 0 .$$

Sasakian manifolds: Let M be a $(2n + 1)$ – dimensional contact metric manifold with contact metric structure (ϕ, ξ, η, g) . If the contact metric structure of M is normal, then M is said to have a Sasakian structure (or normal contact metric structure) and M is called a Sasakian manifold (or normal contact metric manifold). An almost contact metric structure (ϕ, ξ, η, g) on M is a Sasakian structure if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \quad (\text{Blair, 1976}).$$

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold with the structure (ϕ, ξ, η, g) . Then the following relations hold (Blair, 1976).

$$\phi^2 X = -X + \eta(X)\xi \tag{8}$$

$$\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0 \tag{9}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{10}$$

$$R(\xi, X)Y = (\nabla \times \phi)(Y) = g(X, Y)\xi - \eta(Y)X \tag{11}$$

$$\nabla_X \xi = -\phi X, \quad (\nabla_X \eta)(Y) = g(X, \phi Y) \tag{12}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{13}$$

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi \tag{14}$$

$$\eta(R(X, Y)Z) = \phi(Y, Z)\eta(X) - g(X, Z)\eta(Y) \quad (15)$$

$$S(X, \xi) = 2n\eta(X) \quad (16)$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n\eta(X)\eta(Y) \quad (17)$$

for all vector fields X, Y, Z and where ∇ denotes the operator of covariant differentiation with respect to g , ϕ is a skew-symmetric tensor field of type $(1, 1)$, S is the Ricci tensor of type $(0, 2)$ and R is the Riemannian curvature tensor of the manifold.

Conharmonically flat Sasakian manifold: The conharmonic curvature tensor (\bar{C}) of type $(1, 3)$ on a Riemannian manifold (M, g) of dimension $(2n + 1)$ is defined by

$$\bar{C}(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \quad (18)$$

for all $X, Y, Z \in \chi(M)$, where Q is the Ricci operator.

If \bar{C} vanishes identically, then we see that the manifold is conharmonically flat.

Let M be a $(2n + 1)$ – dimensional $(n > 1)$ conharmonically flat Sasakian manifold. Then

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= \frac{1}{2n+1} [S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \\ &+ g(Y, Z)S(X, W) - g(X, Z)S(Y, W)], \end{aligned} \quad (19)$$

where $\tilde{R}(X, Y, Z, W) = g(R(X, Y, Z), W)$.

In (19), putting $Z = \xi$ and in virtue of (9), (13), (16) we get

$$\begin{aligned} g(X, Y)\eta(Y) - g(Y, W)\eta(X) &= \frac{-1}{2n-1} [2ng(Y, W)\eta(X) - 2ng(X, W)\eta(Y) \\ &+ S(Y, W)\eta(X) - S(X, W)\eta(Y)] \end{aligned} \quad (20)$$

Again, putting $X = \xi$ in (20) and using (9), (16) we obtain

$$S(Y, W) = Ag(Y, W) + B\eta(Y)\eta(W) \quad (21)$$

where

$$A = \frac{-4n^2 + 2n + 1}{2n - 1},$$

$$B = \frac{8n^2 - 4n - 1}{2n - 1}$$

and $A + B = 2n$.

Conharmonically ϕ -symmetric Sasakian manifolds: A $(2n + 1)$ – dimensional Sasakian manifold is called conharmonically ϕ -symmetric if

$$\phi^2(\nabla_W \bar{C})(X, Y, Z) = 0 \quad (22)$$

for all vector field X, Y, Z, W orthogonal to ξ , the notion of locally ϕ -symmetric Sasakian manifolds was introduced by T. Takahashii (Takahashi, 1977).

Let M be a $(2n + 1)$ – dimensional Sasakian manifold. Then

$$\phi^2(\nabla_W \bar{C})(X, Y)Z = 0$$

implies

$$(\nabla_W \bar{C})(X, Y, Z) = \eta((\nabla_W \bar{C})(X, Y)Z)\xi$$

From (18)

$$\begin{aligned} (\nabla_W \bar{C})(X, Y)Z &= (\nabla_W R)(X, Y)Z - \frac{1}{2n-1} [(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y \\ &+ g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y] \end{aligned} \tag{23}$$

From (23) it follows that

$$\eta((\nabla_W \bar{C})(X, Y)Z) = \eta((\nabla_W R)(X, Y)Z) \tag{24}$$

since X, Y are orthogonal to ξ .

From (22), (23) and (24), it follows that

$$\begin{aligned} (\nabla_W R)(X, Y)Z - \frac{1}{2n-1} [(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y \\ + g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y] &= \eta((\nabla_W R)(X, Y)Z)\xi \\ \text{or, } -(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi & \\ = -\frac{1}{2n-1} [(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y + g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y] & \end{aligned} \tag{25}$$

Taking the inner product of the above equation with respect to U , we get

$$\begin{aligned} -g((\nabla_W R)(X, Y)Z, U) &= -\frac{1}{2n-1} [(\nabla_W S)(Y, Z)g(X, U) - (\nabla_W S)(X, Z)g(Y, U) \\ + g(Y, Z)g((\nabla_W Q)X, U) - g(X, Z)g((\nabla_W Q)Y, U)] & \end{aligned} \tag{26}$$

Putting $Z = \xi$, we get from (26)

$$-g((\nabla_W R)(X, Y)\xi, U) = -\frac{1}{2n-1} [(\nabla_W S)(Y, \xi)g(X, U) - (\nabla_W S)(X, \xi)g(Y, U)] \tag{27}$$

In (27), putting $X = U = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold we get, $(\nabla_W S)(Y, \xi) = 0$. (28)

Now, we know that

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi),$$

using (12) and (16) in the above relation, we obtain

$$(\nabla_W S)(Y, \xi) = 2n g(W, \phi Y) + S(Y, \phi W) \tag{29}$$

In view of (28) and (29), we get,

$$S(Y, \phi W) = -2n g(W, \phi Y)$$

$$\text{or, } S(Y, \phi W) = 2n g(Y, \phi W).$$

Putting $\phi W = X$, we get

$$S(X, Y) = 2n g(X, Y) \tag{30}$$

Conharmonically semi symmetric Sasakian manifold: A $(2n + 1)$ – dimensional Sasakian manifold is called conharmonically semi symmetric if

$$R.\bar{C} = 0 \tag{31}$$

where \bar{C} is the conharmonic curvature tensor of type (1, 3) and R is the Riemannian curvature tensor of type (1, 3).

Let M be a $(2n + 1)$ – dimensional Sasakian manifold.

If it is conharmonically semi symmetric, then

$$R.\bar{C} = 0 .$$

This implies

$$R(X, Y)\bar{C}(U, V)W - \bar{C}(R(X, Y)U, V)W - \bar{C}(U, R(X, Y)V)W - \bar{C}(U, V)R(X, Y)W = 0 \quad (32)$$

Now from (18), we get

$$\eta(\bar{C}(X, Y)Z) = \left[1 - \frac{2n}{2n-1}\right] [g(Y, Z)\eta(X) - g(X, Y)\eta(Y)] - \frac{1}{2n-1} [S(Y, Z)\eta(X) - S(X, Z)\eta(Y)] \quad (33)$$

Putting $Z = \xi$, we get from (33)

$$\eta(\bar{C}(X, Y)\xi) = 0 \quad (34)$$

For $X = \xi$ (33) gives

$$\eta(\bar{C}(\xi, Y)Z) = \left[1 - \frac{2n}{2n-1}\right] [g(Y, Z) - \eta(Y)\eta(Z)] - \left[\frac{1}{2n-1}\right] [S(Y, Z) - 2n\eta(Y)\eta(Z)] \quad (35)$$

Now (32) implies

$$\begin{aligned} &\bar{C}(U, V, W, Y) - \eta(Y)\eta(\bar{C}(U, V)W) + \eta(U)\eta(\bar{C}(Y, V)W) + \eta(V)\eta(\bar{C}(V, Y)W) \\ &+ \eta(W)\eta(\bar{C}(U, V)Y) - g(Y, U)\eta(\bar{C}(\xi, V)W) \\ &- g(Y, V)\eta(C^2(U, \xi)W) - g(Y, W)\eta(\bar{C}(U, V)\xi) = 0 \end{aligned} \quad (36)$$

where $\bar{C}(U, V, W, Y) = g(\bar{C}(U, V)W, Y)$.

In (36), putting $Y = U$ and using (33) and (34) we obtain

$$\bar{C}(U, V, W, U) - \eta(W)\eta(\bar{C}(U, V)W) - g(U, V)\eta(\bar{C}(\xi, V)W) - g(U, V)\eta(\bar{C}(U, \xi)W) = 0 \quad (37)$$

Putting $u = e_i$, where $\{e_i\}$, $1 \leq i \leq 2n+1$, is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i , and using (33) and (35), we get from (37)

$$S(V, W) = Ag(V, W) + B\eta(V)\eta(W) \quad (38)$$

where

$$A = 4n - 2, \\ B = \frac{-n(2n+1)}{n-1}.$$

Results

From Conharmonically flat Sasakian manifold and (18-21), we are in a position to state the following:

Theorem 1: A $(2n + 1)$ – dimensional $(n > 1)$ conharmonically flat Sasakian manifold is an η -Einstein manifold.

Considering Conharmonically ϕ -symmetric Sasakian manifolds and (22-30), we easily get

Theorem 2: A locally conharmonically ϕ -symmetric Sasakian manifold is an Einstein manifold.

Putting $X = Y = e_i$, where $\{e_i\}$, $1 \leq i \leq 2n+1$, is an orthonormal basis of the tangent space at each point of the manifold, we get,

$r = \text{constant}$, where r is the scalar curvature of the manifold. Thus we obtain the following:

Theorem 3: In a $(2n + 1)$ – dimensional conharmonically ϕ -symmetric Sasakian manifold the scalar curvature is constant.

From Conharmonically semi symmetric Sasakian manifold and (31-38) we can state

Theorem 4: A $(2n + 1)$ – dimensional ($n > 1$) conharmonically semi symmetric Sasakian manifold is an η -Einstein manifold.

Contracting (38) we get

Corollary 1: In a $(2n + 1)$ – dimensional ($n > 1$) conharmonically semi symmetric Sasakian manifold the scalar curvature is constant.

Discussion

Sasakian manifold has an wide applications in super gravity and string theory (Friedrich and Ivanov, 2002). In (Friedrich and Ivanov, 2002), the authors constructed an example of a 5-dimensional Sasakian manifold which satisfies basic equations of string theory, that is,

$$\delta(T) = 0, \quad \nabla \psi = 0, \quad T\psi = \mu\psi,$$

where T is the skew symmetric torsion, ψ is spinor field and μ is a function. If scalar curvature is constant of a Sasakian manifold, then (Friedrich and Ivanov, 2002) it satisfies the above equation. We see that in the earlier sections we obtained scalar curvature constant. Hence the conharmonic curvature tensor has a vital role in super gravity and string theory.

Here we like to construct an example of a three-dimensional conhermonically ϕ -symmetric Sasakian manifold.

Let us consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3, (x, y, z) \neq (0, 0, 0)\}$,

where (x, y, z) are the standard coordinates in \tilde{R} . The vector fields

$$e_1 = 2\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial z}, \quad e_2 = -\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M . Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any Z belongs to $\chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi e_1 = -e_2$, $\phi e_2 = e_1$, $\phi e_3 = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and R be the curvature tensor of g . Then we have

$$[e_1, e_2] = -2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)$$

which is known as Koszul's formula. Taking $e_3 = \xi$ and using the above formula for Riemannian metric g , it can be easily calculated that

$$\begin{aligned} \nabla_{e_1} e_3 &= e_2, & \nabla_{e_1} e_2 &= -e_3, & \nabla_{e_1} e_1 &= 0, \\ \nabla_{e_2} e_3 &= -e_1, & \nabla_{e_2} e_2 &= 0, & \nabla_{e_2} e_1 &= e_3, \\ \nabla_{e_3} e_3 &= 0, & \nabla_{e_3} e_2 &= -e_1, & \nabla_{e_3} e_1 &= e_2. \end{aligned}$$

We see the (ϕ, ξ, η, g) structure satisfies the formula $\nabla_X \xi = -\phi X$. Hence $M(\phi, \xi, \eta, g)$ is a three-dimensional Sasakian manifold.

Using the above relations we obtain the components of the curvature tensor as follows:

$$\begin{aligned} R(e_1, e_2)e_3 &= 0, & R(e_2, e_3)e_3 &= -\frac{1}{2}e_2, & R(e_1, e_3)e_3 &= -\frac{1}{2}e_1, \\ R(e_1, e_2)e_2 &= -\frac{3}{2}e_1, & R(e_2, e_3)e_2 &= \frac{1}{2}e_3, & R(e_1, e_3)e_2 &= 0, \\ R(e_1, e_2)e_1 &= -\frac{3}{2}e_2, & R(e_2, e_3)e_1 &= 0, & R(e_1, e_3)e_1 &= \frac{1}{2}e_3. \end{aligned}$$

From

$$(\nabla_{e_1} R)(e_1, e_2) = (\nabla_{e_2} R)(e_1, e_2)e_2 = -e_3, \quad (\nabla_{e_2} R)(e_1, e_2)e_1 = (\nabla_{e_1} R)(e_1, e_2)e_2 = 0,$$

We see that

$$\begin{aligned} S(e_1, e_2) &= g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = 1 \\ S(e_2, e_2) &= g(R(e_2, e_1)e_1, e_2) + g(R(e_2, e_3)e_3, e_2) = 1 \\ S(e_3, e_3) &= g(R(e_3, e_1)e_1, e_3) + g(R(e_3, e_2)e_2, e_3) = -1 \end{aligned}$$

From the values of $R(X, Y)Z$ and $S(X, Y)$ it follows that $\bar{C}(X, Y)Z$ is a constant. Hence the manifold satisfies $\phi^2(\nabla_W C)(X, Y)Z = 0$. Hence the manifold is conharmonically ϕ -symmetric. Also its scalar curvature is 1, a constant. Thus we see that the results obtained in this example agrees with the results obtained earlier.

Conclusion

The theorem and corollary obtained from this study can be applied to non-linear system and different branches of physics as well.

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