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RESEARCH ARTICLE PHYSICAL SCIENCE

NUMERICAL APPROACH FOR SOLVING SYSTEM OF NONLINEAR EQUATIONS BY ITERATIVE METHODS

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Abstract: This paper deals with solving a system of nonlinear equations by three well-known methods. The numerical solution of *n* nonlinear equations in *n* variables using the methods of Newton's, Broyden's and Brown's are discussed. Some modifications of Newton's and Broyden's methods are added to overcome some difficulties. These three methods are compared both in analytical and numerical prospects. The implementations are compared by a set of problems. Numerical results are obtained by using FORTRAN programming.

Key words: Nonlinear equation, iterative method

Introduction

One of the fundamental problems of mathematics is that of solving equation of the form $f(x)=0$ where f is a real valued function of real variable. Any value x^* satisfying the equation $f(x)=0$ is called a root of the equation. Most equations that arise in physical and engineering problems are nonlinear and are rarely of the form, which allows the root to be determined exactly. Consequently, numerical techniques must be used to find them.

We consider the system of nonlinear equations of n variables of the form, $\mathbf{F}(\mathbf{x}) = 0$, where $\mathbf{F}$ is a function from $\Re^n$ to $\Re^n$ and $\mathbf{x} \in \Re^n$. The one-point iterative methods for the single equation of one variable can be extended to a system of nonlinear equations. This paper deals with solving nonlinear system of equations by three iterative methods namely Newton's, Broyden's and Brown's method.

Newton did not publish an extensive discussion of his method but he solved a cubic polynomial by this method. Broyden introduced a variation of Newton's method incorporating the secant method known as Broyden's method. Brown (1971) proposed a new method in 1969 like Newton's. Later this method is named Brown's method. In this research work we compare these three methods analytically as well as numerically to sort out which one is more efficient under certain conditions.

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Materials and Methods

The general form of a system of nonlinear equations is-

$$\left.\begin{array}{l} f_1(x_1, x_2, \ldots\ldots x_n) = 0 \\ f_2(x_1, x_2, \ldots\ldots x_n) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(x_1, x_2, \ldots\ldots x_n) = 0 \end{array}\right\} \qquad 2.1$$

where each function f_i is a mapping from $\Re^n$ to $\Re$. The system of equations can alternatively be represented by a function $\mathbf{F}: \Re^n \to \Re^n$ in the form $\mathbf{F}(\mathbf{x})=0$, where $\mathbf{x} = (x_1, x_2, \ldots x_n)^t \in \Re^n$ and $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots\ldots f_n(\mathbf{x}))^t$. The functions $f_1, f_2, \ldots f_n$ are called the component functions of **F**.

The best-known method for solving the system of nonlinear equations is Newton's method. Sometimes this method is modified so as to improve its computational efficiency. Quasi-Newton method is one of the variations of Newton's method. Browns method is much more similar to the Gaussian elimination method in n dimensional linear system. To analyze the algorithms for solving $\mathbf{F}(\mathbf{x}) = 0$ the function **F** is assumed to have the following properties:

a) F is continuously differentiable in an open convex set D.

b) There is an $\mathbf{x}^*$ in D such that $F(\mathbf{x}^*)=0$ and $J(\mathbf{x}^*)$ is nonsingular (A)

where $J(\mathbf{x}^*)$ denotes the Jacobian matrix $\left(\dfrac{\partial f_i(\mathbf{x})}{\partial x_j}\right)$ evaluated at the point $\mathbf{x}^*$ so that the conditions (A) guarantees that $\mathbf{x}^*$ is a locally unique solution to the equation $\mathbf{F}(\mathbf{x})=0$. In addition to the conditions (A) sometimes we need a stronger requirement that $J(\mathbf{x})$ satisfies the Lipschitz condition at $\mathbf{x}^*$ i.e. there is a constant K such that

$$\left\|J(\mathbf{x}) - J(\mathbf{x}^*)\right\| \le K\left\|\mathbf{x} - \mathbf{x}^*\right\| \; ; \qquad \mathbf{x} \in D \qquad \text{(B)}$$

Newton's method: For solving n-dimensional system of nonlinear equations $\mathbf{F}(\mathbf{x})=0$, where $\mathbf{x} = (x_1, x_2, \ldots\ldots x_n)^t$, by Newton's method, we have the iteration formula

$$\mathbf{x}_{k+1} = \mathbf{x}_k - J_k^{-1}\mathbf{F}(\mathbf{x}_k)\,; \quad k = 0,1,2,3\ldots\ldots \qquad 2.1.1$$

where $\mathbf{x}_k$ denote the k-th approximation of the solution and J_k denote the Jacobian of the system at the point $\mathbf{x}_k$ It can be shown that the sequence $\{\mathbf{x}_k\}$ obtained from Newton's method converge to the exact solution quadratically.

If the Jacobian is relatively easy to obtain, Newton's method is very attractive. Sometimes exact evaluation of partial derivatives is inconvenient and in many applications may be impossible. In that case we may use finite difference approximations to the partial derivatives. The partial derivatives of a function of n-variables is given by

$$\frac{\partial f(\mathbf{x})}{\partial x_j} \approx \frac{f(\mathbf{x} + \mathbf{e}_j h_j) - f(\mathbf{x})}{h_j} \qquad 2.1.2$$

where $\mathbf{x} = (x_1, x_2, \ldots\ldots x_n)^t$, and h_j is the j-th component of $\mathbf{h} = (h_1, h_2, \ldots\ldots h_n)^t$, and $\mathbf{e}_j$ is a vector in $\Re^n$ whose only non zero component is a one in the j-th coordinate . Thus we can approximate Jacobian $J(\mathbf{x})$ by

$$J(\mathbf{x})=\left[\frac{\partial f_i(\mathbf{x})}{\partial x_j}\right]_{i,j}=\frac{f_i\left(\mathbf{x}+\mathbf{e}_j h_j\right)-f_i(\mathbf{x})}{h_j} \qquad 2.1.3$$

We use this Jacobian in Newton's method and call this method, the finite difference Newton's method. It should be noted that **h** is some suitably chosen vector, which can be updated at each iteration.

Finally we note that Newton's method and finite difference Newton's method require the solution of a system of liner equations and therefore $O(n^3)$ arithmetic operations per iteration. For some problem, the solution of these linear systems is the most expansive part of the iteration. In these cases one should consider the Jacobian matrix fixed for a given number of iterations since for each such iteration, this expanse would be reduced to $O(n^2)$.

Broyden's method: We have seen that Newton's method requires a Jacobian matrix to be solved at each iteration which requires n^2 partial derivatives. The total computational efforts, in one iteration, for Newton's method is consequently at least n^2+n scalar functional evaluations together with order of n^3 arithmetic operations to solve the linear system. This amount of computation is prohibited except for relatively small values of n and easily evaluated scalar function. So we consider a generalization of secant method to system of nonlinear equations. In particular this technique is known as Broyden's method. Here we consider the Broyden's method together with the modification done by Dennis and Moré (1974).

In Broyden's method we assume that $\mathbf{F}:\Re^n \to \Re^n$ satisfies the conditions (A). We want to get a good approximation of $J(\mathbf{x}_{k+1})$, where $\mathbf{x}_{k+1}$ is the $(k+1)$-th approximation to the root. By using the secant method formula for a function of several variables to approximate the Jacobian we get

$$J\left(\mathbf{x}_{k+1}\right)=\frac{\mathbf{F}\left(\mathbf{x}_k\right)-\mathbf{F}\left(\mathbf{x}_{k+1}\right)}{\mathbf{x}_k-\mathbf{x}_{k+1}} \qquad 2.2.1$$

From (2.2.1) it is clear that the degree of approximation of $J(\mathbf{x}_{k+1})$ increases as $\left\|\mathbf{x}_k-\mathbf{x}_{k+1}\right\|$ decreases. We do not know about the term $\mathbf{x}_k$ - $\mathbf{x}_{k+1}$ and the corresponding quotient since it is a vector. So we try to find the matrix $J_{k+1}=J(\mathbf{x}_{k+1})$ which satisfies the equation

$$\mathbf{F}\left(\mathbf{x}_k\right)=\mathbf{F}\left(\mathbf{x}_{k+1}\right)+J_{k+1}\left(\mathbf{x}_k-\mathbf{x}_{k+1}\right). \qquad 2.2.2$$

Now let $\mathbf{s}_k=\mathbf{x}_{k+1}-\mathbf{x}_k$. Then the formula for finding J_{k+1} from J_k is given by the equation

$$J_{k+1}=J_k+\frac{\left[\mathbf{y}-J_k\mathbf{s}_k\right]\mathbf{s}_k^t}{\left\langle \mathbf{s}_k,\mathbf{s}_k\right\rangle} \qquad 2.2.3$$

Equation (2.2.3) is central to the development of the Quasi-Newton methods, and therefore it has often been called the 'Quasi-Newton equation'. In fact, it also plays a role in a second derivation of Broyden's update. Now we use the equation (2.2.3) to define the most basic form of Broyden's method. This method is defined by

$$\mathbf{x}_{k+1}=\mathbf{x}_k-J_k^{-1}\mathbf{F}\left(\mathbf{x}_k\right); \qquad k=0,1,2,3...... \qquad 2.2.4$$

where the matrices $J_k \in L(\Re^n)$ are generated by

$$J_{k+1}=J_k+\frac{\left[\mathbf{y}_k-J_k\mathbf{s}_k\right]\mathbf{s}_k^t}{\left\langle \mathbf{s}_k,\mathbf{s}_k\right\rangle}, \quad k=0,1,2,3...... \qquad 2.2.5$$

with $\mathbf{y}_k=\mathbf{F}\left(\mathbf{x}_{k+1}\right)-\mathbf{F}\left(\mathbf{x}_k\right),$ 2.2.6

and $\mathbf{s}_k=\mathbf{x}_{k+1}-\mathbf{x}_k.$ 2.2.7

It is clear that for given $\mathbf{x}_0$ and J_0, Broyden's method can be carried out with n scalar function evaluations per iteration. However, equations (2.2.4) and (2.2.5) indicate that the solution of the linear system $J_k\mathbf{s}_k = -\mathbf{F}(\mathbf{x}_k)$ is still required. To overcome this difficulty we consider a result which is due to Sherman and Morrison (1949). Sherman and Morrison proposed that the inverse of Jacobian can be approximated by the formula

$$J_{k+1}^{-1} = J_k^{-1} + \frac{\left(\mathbf{s}_k - J_k^{-1}\mathbf{y}_k\right)\mathbf{s}_k^t J_k^{-1}}{\left\langle \mathbf{s}_k, J_k^{-1}\mathbf{y}_k \right\rangle}\text{; provided } \left\langle \mathbf{s}_k, J_k^{-1}\mathbf{y}_k \right\rangle \neq 0. \qquad 2.2.8$$

Therefore, Broyden's method can be implemented as $\mathbf{x}_{k+1} = \mathbf{x}_k - J_k^{-1}\mathbf{F}(\mathbf{x}_k)$ where $\left\{J_k^{-1}\right\}$ is generated by (2.2.8). This form of Broyden's method only requires n scalar function evaluation and $O(n^2)$ arithmetic operations per iteration. Dennis and Moré (1977) showed that this method converges superlinearly to the exact solution.

Browns method: In this section we introduce another variation of Newton's method suggested by Brown (1971). This method is a variation of Newton's method incorporating Gaussian elimination in such a way that all the recent information is used at each step of the algorithm. Here the Jacobian matrix is reduced to the lower triangular form using Gaussian elimination method. For solving the system of nonlinear equations $\mathbf{F}(\mathbf{x}) = 0$ the function $\mathbf{F}$ is assumed to have the properties given by (A). Let $\mathbf{x}_k$ be the k-th approximation of the solution of $\mathbf{F}(\mathbf{x}) = 0$. In this method we approximate the forward triangularization of the full Jacobian matrix by working with one row at a time, eliminating one variable for each row treated.

We now express the Brown's method by writing it in terms of iteration function $G = (G_1, G_2, \ldots\ldots\ldots G_n)$. With an initial approximation $\mathbf{x}_0$ we can find the next approximations by

$$\mathbf{x}_{k+1} = G(\mathbf{x}_k); \qquad k = 0,1,2,3\ldots\ldots \qquad 2.3.1$$

The iteration function G, for Brown's method is given by

$$G_i(x_1, x_2, \ldots\ldots\ldots x_n) = x_i - \sum_{j=1}^{i-1}\left[\left(\frac{\partial g_{n-i+1}/\partial x_j}{\partial g_{n-i+1}/\partial x_i}\right)\left(G_j - x_j\right)\right] - \frac{g_{n-i+1}}{\partial g_{n-i+1}/\partial x_i} \quad \text{for } i = 1,2,3\ldots.n \qquad 2.3.2$$

where G_i for $i = 1,2,3\ldots\ldots n$ are the component functions of G, and

$$\left.\begin{aligned}
&g_1 = f_1(x_1, x_2, \ldots\ldots\ldots\ldots x_n)\\
&g_2 = f_2(x_1, x_2, \ldots\ldots\ldots x_{n-1}, b_n)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&g_i = f_i(x_1, x_2, \ldots\ldots x_{n-i+1}, b_{n-i+2}, \ldots. b_n)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&g_{n-i+1} = f_{n-i+1}(x_1, x_2, \ldots\ldots x_i, b_{i+1}, b_{i+2}, \ldots\ldots b_n)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&g_n = f_n(x_1, b_2, b_3, \ldots\ldots\ldots b_n)
\end{aligned}\right\} \qquad 2.3.3$$

The b_{i+1} for i=1, 2, …..n-1, are themselves functions of the x_j and are obtained recursively by successive substitution in the system

$$b_{m+1} = x_{m+1} - \sum_{j=i+1}^{m}\left[\left(\frac{\partial g_{n-m}/\partial x_j}{\partial g_{n-m}/\partial x_{m+1}}\right)\left(b_j - x_j\right)\right] - \frac{g_{n-m}}{\partial g_{n-m}/\partial x_{m+1}} \quad \text{for } m = i, i+1, \ldots. n-1 \qquad 2.3.4$$

For the purpose of completeness, we define $b_1 = x_1$.

Brown and Dennis (1971) showed jointly that the Brown's method converges quadratically. They also showed that this method required only $(n^2\text{-}3n)/2$ component function evaluation per iteration.

Theoretical comparisons

In this section the methods that are discussed earlier will be compared from the different theoretical aspects.

Component versus vector function evaluation: In Brown's method we have to evaluate $f_i(x)$, at each step, in some applications which are almost as expensive as computing $\mathbf{F}(\mathbf{x})$. But Newton's method is much more attractive in the sense that we can directly deal with the Jacobian. In Broyden's method functional evaluation is much easier, since it involves matrix multiplication and addition. The number of function evaluations per iteration of Broyden(1965), Brown(1971) and Newton(1985) method are n, $(n^2\text{-}3n)/2$ and n^2+n respectively.

Structured problem: It is easier to modify Newton's method so that it can take into account any structure of the problem. Suppose, for example, that $J(\mathbf{x})$ is tri-diagonal. In this case Newton's method can be modified so that it only requires $O(n)$ arithmetic operations for vector function evaluation per iteration(Curtis et al. (1974)). Similar arguments can be constructed for Broyden's method. On the other hand, in Brown's method we can modify the method for the same example that reduces the arithmetic operations to $O(n^2)$ per iteration.

Partially linear system: We found that Brown's method is much more efficient than Newton's method for solving partially linear system. In some applications when the Newton's method and Brown's method fail to converge to the analytic solution of the system, Brown's method easily does the job. Powell (1970) pointed out that if only $f_1(x)$ is nonlinear in the system, then Brown's method and finite difference Newton's method with fixed h generate the same iterates.

Results

We present here some problems with numerical solutions obtained by the methods in support to establish our hypothesis and analysis. Computer program evaluated results are discussed for each problem. To calculate the error term we use the l_∞ norm for the vector $\mathbf{x} \in \Re^n$ which is defined by $\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$. Beside this we use $\mathbf{h} = (0.1, 0.1, 0.1)^t$ to approximate the Jacobian in finite difference Newton's method. Though there is no hard rule for selecting initial approximation we choose the value to ensure it lead us to the exact solutions.

Problem 1:

$$x_1^2 - x_2 - 1 = 0$$
$$(x_1 - 2)^2 + (x_2 - 0.5)^2 - 1 = 0$$

Solution: We take the initial approximation $(0.1, 0.1)^t$. The iterative solutions are summarized in the following tables.

Table 1. Solution by Newton's method.

No. of iteration k	Solution $\mathbf{x}_k^t$	$\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$
0	(0.1, 0.2)	
3	(1.06229, 0.12897)	0.19139
5	(1.0673, 0.13909)	0.13718×10^{-2}
7	(1.06734, 0.13923)	0.16618×10^{-4}
10	(1.06735, 0.13923)	0.10089×10^{-7}

Table 2. Solution by Broyden's method.

No. of iteration k	Solution $\mathbf{x}_k^t$	$\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$
0	(0.1, 0.2)	
2	(1.22206, -0.1385)	0.59474
5	(1.1104, 0.35267)	0.79267×10^{-1}
10	(1.06763, 0.14265)	0.1515×10^{-1}
16	(1.06735, 0.13923)	0.29802×10^{-7}

In this particular problem Newton's, Broyden's and Brown's methods give the iterative solution correct up to 7 decimal places within 10, 16, and 8 iterations respectively. Brown method is more efficient than the other two methods. If we chose the initial approximation $(0.1, 0.1)^t$ then the iteration obtained by Brown's method converges to the other solution of the system $x^* \cong (1.546343, 1.391176)^t$. But with this approximation Broyden's method gives divergent result, while Newton's method takes a large number of iterations.

Table 3. Solution by Brown's method.

No. of iteration k	Solution $\mathbf{x}_k^t$	$\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$
0	(0.1, 0.2)	
1	(1.19874, -0.77025)	0.10987×10^{-1}
5	(1.06532, 0.13339)	0.80515×10^{-1}
8	(1.06734, 0.13923)	0.78564×10^{-9}

Problem 2:

$$3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0$$

$$x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0$$

$$e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$$

Solution: We take the initial approximation $(0.1, 0.1, -0.1)^t$. The iterative solutions are summarized in the following tables.

Table 4. Solution by Newton's method.

No. of iteration k	Solution $\mathbf{x}_k^t$	$\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$
0	(0.1, 0.1, -0.1)	
1	(0.49987, 0.1947 $\times 10^{-1}$, -0.52152)	0.42152
4	(0.5, -0.5032 $\times 10^{-8}$, -0.52359)	0.12444×10^{-4}
5	(0.5, -0.5809 $\times 10^{-8}$, -0.52359)	0.77668×10^{-9}

Table 5. Solution by Broyden's method.

No. of iteration k	Solution $\mathbf{x}_k^t$	$\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$
0	(0.1, 0.1, -0.1)	
2	(0.49997, 0.8738 $\times 10^{-2}$, -0.52317)	0.10729×10^{-1}
5	(0.5, 0.1879 $\times 10^{-5}$, -0.52359)	0.39341×10^{-4}
6	(0.5, 0.5017 $\times 10^{-8}$, -0.52359)	0.19292×10^{-6}

This problem is one of the classical problems used in the well-known text of Burden and Faires (1985). We find that the Brown's method gives different result other than Newton's and Broyden's method with the initial approximation $(0.1, 0.1, -0.1)^t$. The results obtained by Newton's and Broyden's method are same as found by Burden and Faires (1985). Observing the error term we can find that in both cases the results are correct up to 6 decimal places. Actually for the Brown's method we get $\|\mathbf{F}(\mathbf{x})\|_\infty \le 10^{-7}$, whereas $\|\mathbf{F}(\mathbf{x})\|_\infty$ does not converges to zero in Newton's and Broyden's method. This indicates that Brown's method gives more accurate result than the others while Newton's and Broyden's method may give misleading results.

Table 6. Solution by Brown's method.

No. of iteration k	Solution $\mathbf{x}_k^t$	$\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$
0	(0.1, 0.1, -0.1)	
1	(0.49999, 0.02733, -0.52292)	0.42292
2	(0.5, 0.02673, -0.52293)	0.59578×10^{-3}
3	(0.49999, 0.02673, -0.52293)	0.64938×10^{-6}

Problem 3:

$$x_1^2 + x_2 - 37 = 0$$

$$x_1 - x_2^2 - 5 = 0$$

$$x_1 + x_2 + x_3 - 3 = 0$$

Table 7. Solution by Newton's method.

No. of iteration k	Solution $\mathbf{x}_k^t$	$\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$
0	(0.5, 0.5, 0.5)	
1	(21, 16.25, -34.25)	0.3475×10^{2}
5	(5.97135, 1.34335, -4.3147)	0.93123
9	(6.0, 1.0, -4.0)	0.47684×10^{-6}

Solution: We take the initial approximation $(0.5, 0.5, 0.5)^t$. The iterative solutions are summarized in the following tables. This problem is partially nonlinear since the system contains one linear equation. As Brown (1971) described his method, in this problem we have to arrange the system in the order of linearity. Hence the linear equation is placed right on the top of the system. Newton's and Broyden's method do not need this sort of arrangement.

Table 8. Solution by Broyden's method.

No. of iteration k	Solution $\mathbf{x}_k^t$	$\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$
0	(0.1, 0.1, -0.1)	
1	(21, 16.25, -34.25)	0.3475×10^2
5	(-6.62013, -1.23792, -10.85805)	0.36267
10	(-5.75307, 4.02139, 4.33169)	0.23123×10^2
15	(-5.90094, 2.26977, 6.63117)	0.84326×10^1
24	(-5.93113, 1.92177, 7.00955)	0.10184×10^3

Table 9. Solution by Brown's method.

No. of iteration k	Solution $\mathbf{x}_k^t$	$\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_\infty$
0	(0.5, 0.5, 0.5)	
1	(21, 16.25, -34.25)	0.3475×10^2
5	(5.97135, 1.34335, -4.3147)	0.93123
9	(6.0, 1.0, -4.0)	0.47684×10^{-6}

This problem is an interesting one as we observe the numerical results taking the initial approximation $(0.5, 0.5, 0.5)^t$. Newton's and Brown's method converge to the exact solution $(6, 1, -4)^t$ with nine iterations, while Broyden's method fails to converge. In this problem Broyden's method shows oscillatory behavior. This is due to the lacking of closeness of the 1st iteration to the exact solution. Another very interesting part is that Newton's and Brown's method give same iteration. Powell (1970) pointed out that if only $f_1(x)$ is nonlinear in the system, then Brown's method and Newton's method with fixed **h** generate the same iterates. Though it is not the present case, still this problem shows that Brown's method and Newton's method can generate same iterations.

Discussion

We have studied Newton's, Broyden's and Brown's methods in this paper, as they are strong tools to solve nonlinear equations. By solving some selected problems, in our findings, it is evident that Brown's method is more effective than the other two methods. Considering the number of functional evaluation per iteration Broyden's method requires less than the other two. On the other hand Newton's method can be fit to any structure of nonlinear system due to its simple working rule. Broyden's method gives less accurate results relative to the other two methods, but its computational effort is amazingly much finer than that of other ones. In many cases, Broyden's method fails to obtain solution whereas Browns method assures to have convergent result (problem 3). The results show that sometimes Newton's method and Brown's method are equivalent, considering the iterations. It is also found that the Brown's method is more effective in the case of partially linear system.

Conclusion

In this paper we use three iterative methods for solving nonlinear system, among which the Brown's method is used in its original form. In recent times few modifications for Brown's method have done by some authors. We have used some modification of Newton's and Broyden's method in this paper. Finally we conclude that among these methods, Broyden's method is very useful in solving the unconstrained optimization problems, since this method approximates the Jacobian at each step by explicit formula. Besides this, all three methods can be used to solve differential equations of higher order.

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