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RESEARCH ARTICLE **PHYSICAL SCIENCE**

ON FOURTH ORDER MORE CRITICALLY DAMPED NON-LINEAR SYSTEMS UNDER SOME CONDITIONS

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Abstract: Krylov-Bogoliubov-Mitropolskii (KBM) method has been extended for solving fourth order more critically damped non-linear systems. For different damping forces, the solutions obtained by the present method show good coincidence with numerical solutions. The method is illustrated by an example.

Key words: Perturbation, asymptotic solutions, more critically damping

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Introduction

Krylov-Bogoliubov-Mitropolskii (KBM) method is a widely spread method to study non-linear systems with small non-linearities. Sattar (1986) has found an asymptotic solution of a second order critically damped nonlinear system. Sattar (1993) also studied third-order over-damped systems. Alam (2002a) studied a third-order critically damped nonlinear system whose unequal eigen-values are in integral multiple. Alam and Sattar (1996) have extended Bogoliubov's asymptotic method to a third order critically damped nonlinear systems. Alam and Sattar (1997) also presented a unified method for obtaining approximate solutions of third order damped, un-damped and over-damped systems. Alam (2002e) has found a solution of third order more critically damped systems. Akbar *el al.* (2003) has found an asymptotic solution of fourth order damped oscillatory nonlinear system based on the work of extended Krylov-Bogoliubov-Mitropolskii method. Akbar *et al.* (2005) also presented a simple technique for obtaining the over damped solutions of an *n*th order nonlinear system.

Islam *et al.* (2006a) has found a new technique for third order critically damped non-linear systems; Islam *et al.* (2006b) also found a new technique for fourth order critically damped non-linear systems. In the present article we develop a method for solving fourth order more critically damped nonlinear systems.

Materials and Methods

Consider a fourth order weakly nonlinear ordinary differential equation

$$x^{(4)} + k_1 \dddot{x} + k_2 \ddot{x} + k_3 \dot{x} + k_4 x = -\varepsilon \, f(x, \dot{x}, \ddot{x}, \dddot{x}) \qquad (1)$$

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where $x^{(4)}$ denote the fourth derivative of *x*, over dots denote first, second and third derivatives with respect to *t*; k_1, k_2, k_3, k_4 are constants, ε is a small parameter and $f(x, \dot{x}, \ddot{x}, \dddot{x})$ is a given nonlinear function. As the equation is of fourth order so it has four real negative eigen values, where three of the eigen values are equal (for more critically damped) say $-\lambda, -\lambda, -\lambda, -\mu$. When $\varepsilon = 0$, the equation becomes linear and the solution of the linear equation of (1) is

$$x(t,0) = (a_0 + b_0\, t + c_0\, t^2)e^{-\lambda t} + d_0\, e^{-\mu t} \qquad (2)$$

where a_0, b_0, c_0, d_0 are constants of integration.

When $\varepsilon \neq 0$, following Shamsul (2002c) a solution of the equation (1) is sought in the form

$$x(t,\varepsilon) = (a + b\,t + c\,t^2)e^{-\lambda t} + d\,e^{-\mu t} + \varepsilon\, u_1(a, b, c, d, t) + \Lambda \qquad (3)$$

where a, b, c and *d* satisfy the first order differential equation

$$\begin{aligned} \dot{a}(t) &= \varepsilon\, A_1(a,b,c,d,t) + \Lambda \\ \dot{b}(t) &= \varepsilon\, B_1(a,b,c,d,t) + \Lambda \\ \dot{c}(t) &= \varepsilon\, C_1(a,b,c,d,t) + \Lambda \\ \dot{d}(t) &= \varepsilon\, D_1(a,b,c,d,t) + \Lambda \end{aligned} \qquad (4)$$

We only consider first few terms in the series expansion of (3) and (4), we evaluate the functions u_i and $A_i, B_i, C_i, D_i,\ i = 1, 2, \ldots\ldots\ldots, n$ such that a, b, c, d appearing in (3) and (4) satisfy the given differential equation (1). In order to determine these unknown functions it is customary in KBM method that the correction terms, $u_i,\ i = 1, 2, \ldots\ldots n$ must exclude terms (known as secular terms) which make them large. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first Murty (1971).

Now differentiating the equation (3) four times with respect to *t*, substituting the value of *x* and the derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(4)}$ in the original equation (1), utilizing the relations presented in (4) and finally equating the coefficients of ε, we obtain

$$\begin{aligned} & e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\left\{\frac{\partial^2 A_1}{\partial t^2} + 3\frac{\partial B_1}{\partial t} + 6C_1 + t\left(\frac{\partial^2 C_1}{\partial t^2} + 6\frac{\partial C_1}{\partial t}\right) + t^2\,\frac{\partial^2 C_1}{\partial t^2}\right\} \\ & + e^{-\mu t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)^3 D_1 + \left(\frac{\partial}{\partial t} + \lambda\right)^3\left(\frac{\partial}{\partial t} + \mu\right)u_1 = -f^{(0)}(a,b,c,d,t) \end{aligned} \qquad (5)$$

where $f^{(0)}(a,b,c,d,t) = f(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})$ and $x_0 = (a_0 + b_0 t + c_0\, t^2)e^{-\lambda t} + d_0\, e^{-\mu t}$

Now we expand $f^{(0)}$ in the Taylor's series of the form

$$\begin{aligned} f^{(0)} = & \sum_{i,j=1}^{\infty} F_0(a,b,c,d)\,e^{-(i\lambda + j\mu)t} + t\sum_{i,j=1}^{\infty} F_1(a,b,c,d)\,e^{-(i\lambda + j\mu)t} + t^2\sum_{i,j=1}^{\infty} F_2(a,b,c,d)\,e^{-(i\lambda + j\mu)t} + \Lambda \\ & + t^n\sum_{i,j=1}^{\infty} F_n(a,b,c,d)\,e^{-(i\lambda + j\mu)t} \end{aligned} \qquad (6)$$

Thus we can write

$$e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\left\{\frac{\partial^2 A_1}{\partial t^2} + 3\frac{\partial B_1}{\partial t} + 6C_1 + t\left(\frac{\partial^2 B_1}{\partial t^2} + 6\frac{\partial C_1}{\partial t}\right) + t^2\,\frac{\partial^2 C_1}{\partial t^2}\right\}$$

$$+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1+\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)u_1=-\left\{\sum_{i,j=1}^{\infty}F_0(a,b,c,d)\,e^{-(i\lambda+j\mu)t}\right.$$

$$\left.+t\sum_{i,j=1}^{\infty}F_1(a,b,c,d)\,e^{-(i\lambda+j\mu)t}+t^2\sum_{i,j=1}^{\infty}F_2(a,b,c,d)\,e^{-(i\lambda+j\mu)t}+\Lambda\right\} \tag{7}$$

We impose the condition that u_1 can not contain the fundamental terms of $f^{(0)}$, therefore (7) can be separated into five equations for unknowns functions u_1 and A_1, B_1, C_1 and D_1 (see also Sattar, 1986; Alam and Sattar, 1996; Alam, 2001, 2002a).

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 C_1}{\partial t^2}=-\sum_{i,j=1}^{\infty}F_2(a,b,c,d)\,e^{-(i\lambda+j\mu)t} \tag{8}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)=-\sum_{i,j=1}^{\infty}F_1(a,b,c,d)\,e^{-(i\lambda+j\mu)t} \tag{9}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1$$

$$=-\sum_{i,j=1}^{\infty}F_2(a,b,c,d)\,e^{-(i\lambda+j\mu)t} \tag{10}$$

and $$\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)u_1=-t^3\sum_{i,j=1}^{\infty}F_3(a,b,c,d)\,e^{-(i\lambda+j\mu)t}-\Lambda \tag{11}$$

Solving the equation (8), we get the value

$$C_1=\frac{-e^{\lambda t}}{\frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)}\sum_{i,j=1}^{\infty}F_2(a,b,c,d)\,e^{-(i\lambda+j\mu)t} \tag{12}$$

The solution of (11) is

$$u_1=\frac{-1}{\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)}\left\{t^3\sum_{i,j=1}^{\infty}F_3(a,b,c,d)\,e^{-(i\lambda+j\mu)t}+\Lambda\right\} \tag{13}$$

Substituting the value of C_1 from (12) into equation (9), we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 B_1}{\partial t^2}=-6e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial}{\partial t}\left(C_1\right)-\sum_{i,j=1}^{\infty}F_1(a,b,c,d)\,e^{-(i\lambda+j\mu)t} \tag{14}$$

Substituting the value of C_1 from equation (12) into equation (14), we get the value of B_1. Now substituting the value of C_1 from (12) and B_1 from (14) into equation (10), we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 A_1}{\partial t^2}+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1$$
$$=-3e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial B_1}{\partial t}-6e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)C_1-\sum_{i,j=1}^{\infty}F_0(a,b,c,d)e^{-(i\lambda+j\mu)t} \tag{15}$$

Now we have only one equation (15) for obtaining the unknown functions A_1 and D_1. So we need to impose some restrictions. In this paper, we have used the restriction that the term $e^{-(i\lambda+j\mu)t}$ balance with A_1 if $i \geq j$ and the term $e^{-(i\lambda+j\mu)t}$ balance with D_1 if $j > i$. This restriction is important, since under this restriction the coefficients of A_1 and D_1 do not become large and also this restriction is useful in the case of strongly more critically damping systems. The restriction is not used in previous paper (Akbar, 2003, 2005; Islam, 2006a,b; Sattar, 1986; Alam and Sattar, 1996; Alam 2001, 2002a, b, d, e, 2003).

Since a, b, c, d are proportional to small parameter, so they are slowly varying functions of time *t* with the period T and as a first approximation, we may consider them as constants. Thus the solutions of the equation (4) is

$$a=a_0+\varepsilon\int_0^t A_1(a_0,b_0,c_0,d_0,t)\,dt$$
$$b=b_0+\varepsilon\int_0^t B_1(a_0,b_0,c_0,d_0,t)\,dt \tag{16}$$
$$c=c_0+\varepsilon\int_0^t C_1(a_0,b_0,c_0,d_0,t)\,dt$$
$$d=d_0+\varepsilon\int_0^t D_1(a_0,b_0,c_0,d_0,t)\,dt$$

Substitute the value of a, b, c, d and u_1 in the equation (3), we get the complete solution of (1).

Thus the determination of the first approximate solution is completed.

Example

As an example of the above procedure consider a fourth order weakly nonlinear system governed by the ordinary differential equation

$$x^{(4)}+k_1\dddot{x}+k_2\ddot{x}+k_3\dot{x}+k_4x=-\varepsilon x^3 \tag{17}$$

Here, $f=x^3$ Therefore,

$$f^{(0)}=-\Big\{a^3e^{-3\lambda t}+3a^2d\,e^{-(2\lambda+\mu)t}+3ad^2\,e^{-(\lambda+2\mu)t}+d^3e^{-3\mu t}$$
$$+t\left(3a^2be^{-3\lambda t}+6abd\,e^{-(2\lambda+\mu)t}+3bd^2\,e^{-(\lambda+2\mu)t}\right)$$
$$+t^2\left(3ab^2e^{-3\lambda t}+3a^2ce^{-3\lambda t}+3db^2\,e^{-(2\lambda+\mu)t}+6acd\,e^{-(2\lambda+\mu)t}+3cd^2\,e^{-(\lambda+2\mu)t}\right)$$
$$+b^3t^3e^{-3\lambda t}+6abct^3e^{-3\lambda t}+3b^2ct^4e^{-3\lambda t}+3ac^2t^4e^{-3\lambda t}+3bc^2t^5e^{-3\lambda t}+c^3t^6e^{-3\lambda t}$$
$$+6bcdt^3e^{-(2\lambda+\mu)t}+3c^2dt^4e^{-(2\lambda+\mu)t}\Big\}$$

For equation (17), the equation (8)-(11) become respectively

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 C_1}{\partial t^2}=-\begin{Bmatrix}3ab^2e^{-3\lambda t}+3a^2ce^{-3\lambda t}+3db^2\,e^{-(2\lambda+\mu)t}\\ +6acd\,e^{-(2\lambda+\mu)t}+3cd^2\,e^{-(\lambda+2\mu)t}\end{Bmatrix} \tag{18}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)=-\left\{3a^2be^{-3\lambda t}+6abd\,e^{-(2\lambda+\mu)t}+3bd^2\,e^{-(\lambda+2\mu)t}\right\} \tag{19}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1$$

$$=-\left\{a^3e^{-3\lambda t}+3a^2d\,e^{-(2\lambda+\mu)t}+3ad^2\,e^{-(\lambda+2\mu)t}+d^3e^{-3\lambda_4 t}\right\} \tag{20}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)u_1=-\Big\{&b^3t^3e^{-3\lambda t}+6abct^3\,e^{-3\lambda t}+3b^2ct^4\,e^{-3\lambda t}+3ac^2\,t^4\,e^{-3\lambda t}\\&+3bc^2\,t^5\,e^{-3\lambda t}+c^3\,t^6\,e^{-3\lambda t}+6bcdt^3\,e^{-(2\lambda+\mu)t}+3c^2\,dt^4\,e^{-(2\lambda+\mu)t}\Big\}\end{aligned} \tag{21}$$

The solution of the equation (18) is

$$C_1=l_1\left(ab^2e^{-2\lambda t}+a^2\,ce^{-2\lambda t}\right)+l_2\left(db^2\,e^{-(\lambda+\mu)t}+2acd\,e^{-(\lambda+\mu)t}\right)+l_3\,cd^2\,e^{-2\mu t} \tag{22}$$

where $l_1=-\dfrac{3}{4\lambda^2(\mu-3\lambda)}$, $l_2=\dfrac{3}{2\lambda(\lambda+\mu)^2}$, $l_3=-\dfrac{3}{4\mu^2(\mu+\lambda)}$

Putting the value of C_1 from the equation (22) in the equation (19), we obtain

$$\begin{aligned}B_1=&\,m_1\left(ab^2+a^2\,c\right)e^{-2\lambda t}+m_2\left(b^2\,d+2acd\right)e^{-(\lambda+\mu)t}+m_3\,cd^2\,e^{-2\mu t}\\&+m_4\,a^2be^{-2\lambda t}+m_5abd\,e^{-(\lambda+\mu)t}+m_6bd^2\,e^{-2\mu t}\end{aligned} \tag{23}$$

where $m_1=-\dfrac{9}{4\lambda^3(\mu-3\lambda)}$, $m_2=\dfrac{9}{\lambda(\lambda+\mu)^3}$, $m_3=\dfrac{9}{4\mu^3(\mu+\lambda)}$,

$m_4=-\dfrac{3}{4\lambda^2(\mu-3\lambda)}$, $m_5=\dfrac{3}{\lambda(\lambda+\mu)^2}$, $m_6=\dfrac{3}{4\mu^2(\mu+\lambda)}$

To separate the equation (20) for determining unknown functions A_1 and D_1, in this paper we impose a restriction (the term $e^{-(i\lambda+j\mu)t}$ balance with A_1 if $i\geq j$ and D_1 if $j>i$). Under this restriction, we obtain

$$\begin{aligned}e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 A_1}{\partial t^2}=&\;6m_1\,\lambda(\mu-3\lambda)\left(ab^2+a^2\,c\right)e^{-3\lambda t}\\&-6\lambda(\lambda+\mu)m_2\left(b^2\,d+2acd\right)e^{-(2\lambda+\mu)t}+6\lambda(\mu-3\lambda)m_4\,a^2be^{-3\lambda t}\\&-6\lambda(\lambda+\mu)m_5abd\,e^{-(2\lambda+\mu)t}-6(\mu-3\lambda)l_1\left(ab^2+a^2\,c\right)e^{-3\lambda t}\\&+12\lambda l_2\left(db^2+2acd\right)e^{-(2\lambda+\mu)t}-a^3e^{-3\lambda t}-3a^2d\,e^{-(2\lambda+\mu)t}\end{aligned} \tag{24}$$

$$e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1=-6\mu(\lambda+\mu)m_3\,cd^2\,e^{-(\lambda+2\mu)t}-6\mu(\lambda+\mu)m_6bd^2\,e^{-(\lambda+2\mu)t}$$

$$+6(\lambda+\mu)l_3\,cd^2\,e^{-(\lambda+2\mu)t}-3ad^2\,e^{-(\lambda+2\mu)t}-d^3e^{-3\mu t} \tag{25}$$

The particular solutions of (24) and (25) respectively become

$$\begin{aligned} A_1 &= n_1\left(a b^2 + a^2 c\right) e^{-2\lambda t} + n_2 \left(b^2 d + 2 a c d\right) e^{-(\lambda+\mu) t} \\ &\quad + n_3\, a^2 b e^{-2\lambda t} + n_4\, a b d\, e^{-(\lambda+\mu) t} + n_5 \left(a b^2 + a^2 c\right) e^{-2\lambda t} \\ &\quad + n_6 \left(d b^2 + 2 a c d\right) e^{-(\lambda+\mu) t} + n_7 a^3 e^{-2\lambda t} + n_8 a^2 d\, e^{-(\lambda+\mu) t} \end{aligned} \tag{26}$$

$$D_1 = p_1\, c d^2\, e^{-(\lambda+\mu) t} + p_2 b d^2\, e^{-(\lambda+\mu) t} + p_3\, a d^2\, e^{-(\lambda+\mu) t} + p_4 d^3 e^{-2\mu t} \tag{27}$$

where $n_1 = -\dfrac{27}{8\lambda^4(\mu - 3\lambda)}$, $n_2 = \dfrac{9}{\lambda(\mu+\lambda)^4}$, $n_3 = -\dfrac{9}{8\lambda^3(\mu - 3\lambda)}$,

$n_4 = \dfrac{9}{\lambda(\mu+\lambda)^3}$, $n_5 = \dfrac{9}{8\lambda^4(\mu - 3\lambda)}$, $n_6 = -\dfrac{9}{\lambda(\mu+\lambda)^4}$,

$n_7 = \dfrac{1}{4\lambda^2(\mu - 3\lambda)}$, $n_8 = \dfrac{3}{2\lambda(\mu+\lambda)^2}$, $p_1 = \dfrac{9}{4\mu^5}$,

$p_2 = \dfrac{9}{16\mu^4}$, $p_3 = \dfrac{3}{8\mu^3}$ $n_4 = \dfrac{1}{(\lambda - 3\mu)^3}$

The solution of the equation (21) for u_1 is

$$\begin{aligned} u_1 &= \left(r_1 t^3 + r_2 t^2 + r_3 t + r_4\right)\left(b^3 + 6 a b c\right) e^{-3\lambda t} + \left(r_5 t^4 + r_6 t^3 + r_7 t^2 + r_8 t + r_9\right) \\ &\quad \times (b^2 c + a c^2) e^{-3\lambda t} + \left(r_{10} t^5 + r_{11} t^4 + r_{12} t^3 + r_{13} t^2 + r_{14} t + r_{15}\right) b c^2 e^{-3\lambda t} \\ &\quad + \left(r_{16} t^6 + r_{17} t^5 + r_{18} t^4 + r_{19} t^3 + r_{20} t^2 + r_{21} t + r_{22}\right) c^3 e^{-3\lambda t} \\ &\quad + \left(r_{23} t^3 + r_{24} t^2 + r_{25} t + r_{26}\right) b c d\, e^{-(\mu+2\lambda) t} \\ &\quad + \left(r_{27} t^4 + r_{28} t^3 + r_{29} t^2 + r_{30} t + r_{31}\right) c^2 d\, e^{-(\mu+2\lambda) t} \end{aligned} \tag{28}$$

where $r_1 = \dfrac{1}{8\lambda^3(\mu - 3\lambda)}$ $r_2 = r_1\left\{-\dfrac{3}{(\mu - 3\lambda)} + \dfrac{9}{2\lambda}\right\}$

$$r_3 = r_1\left\{\frac{6}{(\mu - 3\lambda)^2} - \frac{9}{\lambda(\mu - 3\lambda)} + \frac{9}{\lambda^2}\right\}$$

$$r_4 = r_1\left\{-\frac{6}{(\mu - 3\lambda)^3} + \frac{9}{\lambda(\mu - 3\lambda)^2} - \frac{9}{\lambda^2(\mu - 3\lambda)} + \frac{15}{2\lambda^3}\right\}$$

$r_5 = \dfrac{3}{8\lambda^3(\mu - 3\lambda)}$ $r_6 = r_5 \times \left\{-\dfrac{4}{(\mu - 3\lambda)} + \dfrac{6}{\lambda}\right\}$

$$r_7 = r_5 \times \left\{\frac{12}{(\mu - 3\lambda)^2} - \frac{18}{\lambda(\mu - 3\lambda)} + \frac{18}{\lambda^2}\right\}$$

$$r_8 = r_5 \times \left\{-\frac{24}{(\mu - 3\lambda)^3} + \frac{36}{\lambda(\mu - 3\lambda)^2} - \frac{36}{\lambda^2(\mu - 3\lambda)} + \frac{30}{\lambda^3}\right\}$$

$$r_9 = r_5 \times \left\{\frac{24}{(\mu - 3\lambda)^4} - \frac{36}{\lambda(\mu - 3\lambda)^3} + \frac{36}{\lambda^2(\mu - 3\lambda)^2} - \frac{30}{\lambda^3(\mu - 3\lambda)} + \frac{45}{2\lambda^4}\right\}$$

$r_{10} = \dfrac{3}{8\lambda^3(\mu - 3\lambda)}$ $r_{11} = r_{10} \times \left\{-\dfrac{5}{(\mu - 3\lambda)} + \dfrac{15}{2\lambda}\right\}$

$$r_{12} = r_{10} \times \left\{ \frac{20}{(\mu-3\lambda)^2} - \frac{30}{\lambda(\mu-3\lambda)} + \frac{30}{\lambda^2} \right\}$$

$$r_{13} = r_{10} \times \left\{ -\frac{60}{(\mu-3\lambda)^3} + \frac{90}{\lambda(\mu-3\lambda)^2} - \frac{90}{\lambda^2(\mu-3\lambda)} + \frac{75}{\lambda^3} \right\}$$

$$r_{14} = r_{10} \times \left\{ \frac{120}{(\mu-3\lambda)^4} - \frac{180}{\lambda(\mu-3\lambda)^3} + \frac{180}{\lambda^2(\mu-3\lambda)^2} - \frac{150}{\lambda^3(\mu-3\lambda)} + \frac{225}{2\lambda^4} \right\}$$

$$r_{15} = r_{10} \times \left\{ -\frac{120}{(\mu-3\lambda)^5} + \frac{180}{\lambda(\mu-3\lambda)^4} - \frac{180}{\lambda^2(\mu-3\lambda)^3} + \frac{150}{\lambda^3(\mu-3\lambda)^2} \right.$$
$$\left. - \frac{150}{\lambda^4(\mu-3\lambda)} + \frac{315}{4\lambda^5} \right\}$$

$$r_{16} = \frac{1}{8\lambda^3(\mu-3\lambda)} \qquad r_{17} = r_{16} \times \left\{ -\frac{6}{(\mu-3\lambda)} + \frac{9}{2\lambda} \right\}$$

$$r_{18} = r_{16} \times \left\{ \frac{30}{(\mu-3\lambda)^2} - \frac{45}{\lambda(\mu-3\lambda)} + \frac{45}{\lambda^2} \right\}$$

$$r_{19} = r_{16} \times \left\{ -\frac{120}{(\mu-3\lambda)^3} + \frac{180}{\lambda(\mu-3\lambda)^2} - \frac{180}{\lambda^2(\mu-3\lambda)} + \frac{150}{\lambda^3} \right\}$$

$$r_{20} = r_{16} \times \left\{ \frac{360}{(\mu-3\lambda)^4} - \frac{540}{\lambda(\mu-3\lambda)^3} + \frac{540}{\lambda^2(\mu-3\lambda)^2} - \frac{450}{\lambda^3(\mu-3\lambda)} + \frac{675}{2\lambda^4} \right\}$$

$$r_{21} = r_{16} \times \left\{ -\frac{720}{(\mu-3\lambda)^5} + \frac{1080}{\lambda(\mu-3\lambda)^4} - \frac{1080}{\lambda^2(\mu-3\lambda)^3} + \frac{900}{\lambda^3(\mu-3\lambda)^2} \right.$$
$$\left. - \frac{675}{\lambda^4(\mu-3\lambda)} + \frac{945}{2\lambda^5} \right\}$$

$$r_{22} = r_{16} \times \left\{ \frac{720}{(\mu-3\lambda)^6} - \frac{1080}{\lambda(\mu-3\lambda)^5} + \frac{1080}{\lambda^2(\mu-3\lambda)^4} - \frac{900}{\lambda^3(\mu-3\lambda)^3} \right.$$
$$\left. + \frac{675}{\lambda^4(\mu-3\lambda)^2} - \frac{945}{2\lambda^5(\mu-3\lambda)} + \frac{315}{\lambda^5} \right\}$$

$$r_{23} = -\frac{3}{2\lambda(\lambda+\mu)^3} \qquad r_{24} = r_{23} \left\{ \frac{3}{2\lambda} + \frac{9}{(\lambda+\mu)} \right\}$$

$$r_{25} = r_{23} \left\{ \frac{3}{2\lambda^2} + \frac{9}{\lambda(\lambda+\mu)} + \frac{9}{(\lambda+\mu)^2} \right\}$$

$$r_{26} = r_{23} \left\{ \frac{3}{4\lambda^3} + \frac{9}{2\lambda^2(\lambda+\mu)} + \frac{9}{\lambda(\lambda+\mu)^2} + \frac{60}{(\lambda+\mu)^3} \right\}$$

$$r_{27} = -\frac{3}{2\lambda(\lambda+\mu)^3} \qquad r_{28} = r_{27} \times \left\{ \frac{2}{\lambda} + \frac{12}{(\lambda+\mu)} \right\}$$

$$r_{29} = r_{27} \times \left\{ \frac{3}{\lambda^2} + \frac{18}{\lambda(\lambda+\mu)} + \frac{72}{(\lambda+\mu)^2} \right\}$$

$$r_{30} = r_{28} \times \left\{ \frac{3}{\lambda^3} + \frac{18}{\lambda^2 (\lambda+\mu)} + \frac{72}{\lambda (\lambda+\mu)^2} + \frac{240}{(\lambda+\mu)^3} \right\}$$

$$r_{31} = r_{28} \times \left\{ \frac{3}{2\lambda^4} + \frac{9}{\lambda^3 (\lambda+\mu)} + \frac{36}{\lambda^2 (\lambda+\mu)^2} + \frac{120}{\lambda (\lambda+\mu)^3} + \frac{360}{(\lambda+\mu)^4} \right\}$$

Substituting the values of A_1, B_1, C_1, D_1 from the equations (26), (23), (22) and (27) into equation (16), we obtain

$$\begin{aligned} a = a_0 + \varepsilon \Bigg\{ & \frac{n_1 \left(a_0\, b_0^2 + a_0^2\, c_0 \right)\left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_2 \left(b_0^2\, d_0 + 2\, a_0\, c_0\, d_0 \right)\left(1 - e^{-(\lambda+\mu) t}\right)}{(\lambda+\mu)} \\ & + \frac{n_3\, a_0^2\, b_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_4\, a_0\, b_0\, d_0\, e^{-(\lambda+\mu) t}}{(\lambda+\mu)} + \frac{n_5 \left(a_0\, b_0^2 + a_0^2\, c_0 \right)\left(1 - e^{-2\lambda t}\right)}{2\lambda} \\ & + \frac{n_6 \left(b_0^2\, d_0 + 2\, a_0\, c_0\, d_0 \right)\left(1 - e^{-(\lambda+\mu) t}\right)}{(\lambda+\mu)} + \frac{n_7\, a_0^3 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_8\, a_0^2\, d_0 \left(1 - e^{-(\lambda+\mu) t}\right)}{(\lambda+\mu)} \Bigg\} \end{aligned} \tag{29}$$

$$\begin{aligned} b = b_0 + \varepsilon \Bigg\{ & \frac{m_1 \left(a_0\, b_0^2 + a_0^2\, c_0 \right)\left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{m_2 \left(b_0^2\, d_0 + 2\, a_0\, c_0\, d_0 \right)\left(1 - e^{-(\lambda+\mu) t}\right)}{(\lambda+\mu)} \\ & + \frac{m_3\, c_0\, d_0^2 \left(1 - e^{-2\mu t}\right)}{2\mu} + \frac{m_4\, a_0^2 b_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{m_5 a_0\, b_0\, d_0 \left(1 - e^{-(\lambda+\mu) t}\right)}{(\lambda+\mu)} \\ & + \frac{m_6 b_0\, d_0^2 \left(1 - e^{-2\mu t}\right)}{2\mu} \Bigg\} \end{aligned} \tag{30}$$

$$c = c_0 + \varepsilon \left\{ \frac{l_1 \left(a_0\, b_0^2 + a_0^2\, c_0 \right)\left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{l_2 \left(b_0^2\, d_0 + 2\, a_0\, c_0\, d_0 \right) e^{-(\lambda+\mu) t}}{(\lambda+\mu)} + \frac{l_3\, c_0\, d_0^2\, e^{-2\mu t}}{2\mu} \right\} \tag{31}$$

$$d = d_0 + \varepsilon \left\{ \frac{p_1\, c_0\, d_0^2\, e^{-(\lambda+\mu) t}}{(\lambda+\mu)} + \frac{p_2\, b_0\, d_0^2\, e^{-(\lambda+\mu) t}}{(\lambda+\mu)} + \frac{p_3\, a_0\, d_0^2\, e^{-(\lambda+\mu) t}}{(\lambda+\mu)} + \frac{p_4\, d_0^3\, e^{-2\mu t}}{2\mu} \right\} \tag{32}$$

Therefore we obtain the first approximate solution of the equation (17) as

$$x(t,\varepsilon) = (a + b\,t + c\,t^2) e^{-\lambda t} + d\, e^{-\mu t} + \varepsilon\, u_1 (a, b, c, d, t) \tag{33}$$

where a, b, c, d are given by the equations (29)-(32) and u_1 given by (28).

Results

It is usual to compare the perturbation solutions to the numerical solutions to test the accuracy of the approximate solutions. With regard to such a comparison concerning the presented KBM method of this paper, we refer to work of Murty *et al.* (1969). In the present paper, we have compared the solutions obtained by (33) to those obtained by fourth order Runge-Kutta method for

Table 1. Comparison between perturbation and numerical results when, $a_0 = 0.7,\ b_0 = 0.0,\ c_0 = 0.0,\ d_0 = 0.1$ and $\varepsilon = 0.1$.

t	x_1	x_1^*	$E_1\%$	x_2	x_2^*	$E_2\%$
0.0	0.800000	0.800000	0.00000	0.800000	0.800000	0.00000
0.5	0.113114	0.112129	0.01337	0.096097	0.096106	0.00936
1.0	0.015841	0.015853	0.07569	0.012756	0.012762	0.04701
1.5	0.002260	0.002264	0.17667	0.001931	0.001933	0.10346
2.0	0.000326	0.000327	0.30581	0.000334	0.000335	0.29850
3.0	0.000007	0.000007	0.00000	0.000013	0.000013	0.00000
3.5	0.000001	0.000001	0.00000	0.000003	0.000003	0.00000
4.0	0.000000	0.000000	0.00000	0.000001	0.000001	0.00000
4.5	0.000000	0.000000	0.00000	0.000000	0.000000	0.00000
5.0	0.000000	0.000000	0.00000	0.000000	0.000000	0.00000

x_1 is computed by (33) using (i) $\lambda = 4,\ \mu = 3.5$; x_2 is computed by (33) using (ii) $\lambda = 4.5,\ \mu = 3$; x_1^* and x_2^* are corresponding numerical solutions.

different values of λ, μ as well as different set of initial conditions.

First of all $x(t,\varepsilon)$ has been computed by (33) in which a, b, c, d are calculated by (29)-(32) with initial conditions $a_0 = 0.7$, $b_0 = 0.0$, $c_0 = 0.0$, $d_0 = 0.1$ for different values of λ and μ (i) $\lambda = 4$, $\mu = 3.5$ (ii) $\lambda = 4.5$, $\mu = 3$ (iii) $\lambda = 4$, $\mu = 2$ (iv) $\lambda = 2.8$, $\mu = 1$ when $\varepsilon = 0.1$. The corresponding numerical solution has been computed by Runge-Kutta method and percentage errors are calculated. All the results are presented in Table 1 and Table 2.

Again $x(t,\varepsilon)$ has been computed by (33) of the same eigen-values (i) $\lambda = 4$, $\mu = 3.5$ (ii) $\lambda = 4.5$, $\mu = 3$ (iii) $\lambda = 4$, $\mu = 2$ (iv) $\lambda = 2.8$, $\mu = 1$ with another set of initial conditions $a_0 = 0.8$, $b_0 = 0.0$, $c_0 = 0.1$, $d_0 = 0.1$ when $\varepsilon = 0.25$. The corresponding numerical solutions are computed by Runge-Kutta method and are presented in Table 3 and Table 4.

Table 2. Comparison between perturbation and numerical results when, $a_0 = 0.7$, $b_0 = 0.0$, $c_0 = 0.0$, $d_0 = 0.1$ and $\varepsilon = 0.1$.

t	x_3	x_3^*	$E_3\%$	x_4	x_4^*	$E_4\%$
0.0	0.800000	0.800000	0.00000	0.800000	0.800000	0.00000
0.5	0.131562	0.131539	0.01748	0.238424	0.233795	1.9799
1.0	0.026370	0.026363	0.02655	0.083581	0.079757	4.7945
1.5	0.006720	0.006717	0.04466	0.035614	0.033047	7.7677
2.0	0.002069	0.002067	0.09675	0.017576	0.016253	9.9858
2.5	0.000706	0.000706	0.00000	0.009923	0.008919	11.256
3.0	0.000252	0.000252	0.00000	0.005791	0.005177	11.860
3.5	0.000092	0.000092	0.00000	0.003457	0.003082	12.167
4.0	0.000034	0.000034	0.00000	0.002083	0.001855	12.291
4.5	0.000012	0.000012	0.00000	0.001260	0.001122	12.299
5.0	0.000005	0.000005	0.00000	0.000763	0.000679	12.371

x_3 is computed by (33) using (iii) $\lambda = 4$, $\mu = 2$; x_4 is computed by (33) using (iv) $\lambda = 2.8$, $\mu = 1$; x_3^* and x_4^* are corresponding numerical solutions.

Table 3. Comparison between perturbation and numerical results when, $a_0 = 0.8$, $b_0 = 0.0$, $c_0 = 0.1$, $d_0 = 0.1$ and $\varepsilon = 0.25$.

t	x_1	x_1^*	$E_1\%$	x_2	x_2^*	$E_2\%$
0.0	0.900000	0.900000	0.00000	0.900000	0.900000	0.00000
0.5	0.129031	0.129070	0.03021	0.109273	0.109301	0.02561
1.0	0.019504	0.019531	0.13824	0.014978	0.014994	0.10670
1.5	0.003066	0.003075	0.29268	0.002311	0.002316	0.21588
2.0	0.000494	0.000496	0.40322	0.000396	0.000397	0.25188
2.5	0.000081	0.000081	0.00000	0.000074	0.000074	0.00000
3.0	0.000013	0.000013	0.00000	0.000015	0.000015	0.00000
3.5	0.000002	0.000002	0.00000	0.000003	0.000003	0.00000
4.0	0.000000	0.000000	0.00000	0.000001	0.000001	0.00000
4.5	0.000000	0.000000	0.00000	0.000000	0.000000	0.00000
5.0	0.000000	0.000000	0.00000	0.000000	0.000000	0.00000

x_1 is computed by (33) using (i) $\lambda = 4$, $\mu = 3.5$; x_2 is computed by (33) using (ii) $\lambda = 4.5$, $\mu = 3$; x_1^* and x_2^* are corresponding numerical solutions.

Table 4. Comparison between perturbation and numerical results when, $a_0 = 0.8$, $b_0 = 0.0$, $c_0 = 0.1$, $d_0 = 0.1$ and $\varepsilon = 0.25$.

t	x_3	x_3^*	$E_3\%$	x_4	x_4^*	$E_4\%$
0.0	0.900000	0.900000	0.00000	0.899989	0.899989	0.00000
0.5	0.148487	0.148479	0.00538	0.269633	0.265360	1.6102
1.0	0.030037	0.030032	0.01664	0.096009	0.092218	4.1109
1.5	0.007526	0.007520	0.07978	0.040649	0.037885	7.2957
2.0	0.002237	0.002232	0.22401	0.019822	0.017957	10.385
2.5	0.000739	0.000737	0.27137	0.010642	0.009438	12.756
3.0	0.000259	0.000258	0.38759	0.006051	0.005295	14.277
3.5	0.000093	0.000093	0.00000	0.003551	0.003083	15.180
4.0	0.000034	0.000034	0.00000	0.002119	0.001832	15.665
4.5	0.000012	0.000012	0.00000	0.001275	0.001100	15.909
5.0	0.000005	0.000005	0.00000	0.000770	0.000664	15.963

x_3 is computed by (33) using (iii) $\lambda = 4$, $\mu = 2$; x_4 is computed by (33) using (iv) $\lambda = 2.8$, $\mu = 1$; x_3^* and x_4^* are corresponding numerical solutions.

Discussion

When $\lambda = 3\mu$ (i.e. $\lambda : \mu = 3:1$) or $\mu = 3\lambda$ (i.e. $\mu : \lambda = 3:1$) the solution (33) is broken-down. From Table 1 we see that when the ratio of λ and μ is $O(1)$, the errors are greater than the errors when the ratio of λ and μ is 1.5. Again from Table 2, we see that, when the ratio of λ and μ is $O(3)$ the errors are greater than the error when the ratio of λ and μ is 2. Thus we see that when the ratio of the eigen-values λ and μ lies between $O(1)$ and $O(3)$, the errors occur much smaller than 1%. When the ratio of the eigen-values λ and μ is $O(1)$, the system undergoes strongly more critically damping. In this case the errors are also smaller than 1%. Therefore the solution is also usable for strongly more critically damping systems.

Conclusion

In the presence of different critically damping effect, a formula has been presented for obtaining the solution of critically-damped non-linear systems governed by the fourth order ordinary differential equation. We see that the solutions obtained by this method show good coincidence with the corresponding numerical solutions. The solutions are also useful for strongly more critically system.

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