



## APPROXIMATION BY HAAR WAVELET AND THEIR ORDER OF APPROXIMATION IN DIFFERENT SPACES

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**Abstract:** Wavelet analysis uses as a new class of orthogonal expansion in  $L_2(\mathfrak{R})$  with regularity-approximation properties. In this study, we approximate any general function in  $L_2(\mathfrak{R})$  by Haar wavelet in different smoothness spaces such as Lebesgue space, Lipschitz continuous space, Sobolev and Besov spaces.

**Key words:** Approximation, wavelet approximation, HAAR wavelet, approximation order, space

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### Introduction

Origin of approximation theory can trace back to Weierstrass approximation theorem proved by Weierstrass in 1885. Since then several celebrated mathematicians notably Picard, Volterra, Lebesgue, Mittag-Leffler, Fejér, Lerch, Landau, de LaVallee Poussin and Bernstein provided alternative proofs of this theorem. Theorems in which the rate of decrease of the best approximation established for a family of functions and different class of polynomials generally called theorems of Jackson type in recognition of the pioneering research work of Jackson between 1911 and 1930. It may remark that Natanson (1949) initiated a systematic study of approximation theory in fifties. The celebrated finite element method developed by engineers in early fifties found close connection with the approximation theory. French mathematician C ea observed in early sixties that error estimation of finite element is nothing but an approximation problem in Sobolev spaces. When wavelet theory appeared in the mid 1980s after that approximation theory come into a new horizon. For details history, one can see Christensen (2004), Cohen (2003), Cohen et al. (1992), DeVore (1998), Islam (2005).

In recent years, there have been many developments and new applications of wavelet analysis to a variety of problems in diverse fields including signal processing, data storage, computer vision, seismology, turbulence, computer graphics, image processing, Galaxies structure, digital communication, pattern recognition, approximation theory, quantum optics, biomedical engineering, sampling theory, matrix theory, differential equations, numerical analysis, statistics etc.

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From the viewpoint of approximation theory and harmonic analysis, the wavelet theory is important on several counts. It provides simple and elegant unconditional wavelet bases for function spaces (Lebesgue, Sobolev, Besov ).

A recent development of approximation theory is approximation of an arbitrary function by wavelet polynomials. There are large examples of wavelet such as Haar wavelet, Mexican-hat wavelet, Shannon wavelet, Daubechies wavelet, Meyer wavelet etc. In this study, we mainly focus on approximation by Haar wavelet. Haar function is the simplest example of wavelet and reflects significant features of the general wavelet. Celebrated wavelet known as Daubechies wavelet is a generalization of Haar wavelet.

In this study we approximate any general function in  $L_2(\mathfrak{R})$  by Haar wavelet and to compare the approximation order in different smoothness spaces.

### Materials and Methods

**Definition of wavelet:** The wavelet means small waves (the sinusoids used in Fourier analysis are big waves) and in brief, a wavelet is an oscillation that decays quickly. Equivalent mathematical conditions are:

$$\int_{\mathfrak{R}} |\psi(t)|^2 dt < \infty \quad (1)$$

$$\int_{\mathfrak{R}} \psi(t) dt = 0 \quad (2)$$

$$\int \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty, \quad (3)$$

where  $\hat{\psi}(\omega)$  is the Fourier Transform of  $\Psi(t)$ .

A function  $\Psi \in L_2(\mathfrak{R})$  is a wavelet if the family of functions  $\psi_{j,k}(t)$  defined by

$$\psi_{j,k}(t) = 2^{\frac{j}{2}} \psi(2^j t - k) \quad (6)$$

where  $j, k \in \mathbb{Z}$  is an orthonormal basis in the Hilbert space  $L_2(\mathfrak{R})$ .

*Definition (Orthogonal basis and Orthonormal basis):* A basis  $\{\psi_n\}_{n=1}^{\infty}$  of Hilbert space H is called orthogonal if

$$\langle \psi_n, \psi_m \rangle = 0 \quad \text{for } n \neq m$$

$$\text{And orthonormal if } \langle \psi_n, \psi_m \rangle = 1 \quad \text{for } n = m$$

A orthogonal basis  $\{\psi_n\}_{n=1}^{\infty}$  is complete in the sense that if  $\langle \psi_n, \psi_m \rangle = 0, \forall n \in \mathbb{N}$ .

### Haar wavelet system

**Haar wavelet:** *Definition (Haar function):* A function defined on the real line  $\mathfrak{R}$  as

$$\psi(t) = \begin{cases} 1 & \text{for } t \in [0, 1/2) \\ -1 & \text{for } t \in [1/2, 1) \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

is known as Haar function.

The Haar function  $\Psi(t)$  is the simplest example of Haar wavelet. The Haar function  $\Psi(t)$  is a wavelet because it satisfies all the conditions of wavelet. Haar wavelet is discontinuous at  $t = 0, 1/2, 1$  and it is very well localized in the time domain.

**Definition of Dyadic interval:** For each pair of  $j, k \in \mathbb{Z}$ , define the intervals  $I_{j,k}$  by  $I_{j,k} = [2^j k, 2^j(k+1))$  which is known as dyadic interval. The collection of all such interval is called dyadic subintervals of  $\mathfrak{R}$ .

**Definition of Haar scaling function:** The family of functions  $\{\varphi_{i,k}(t)\}_{i,k \in \mathbb{Z}} = 2^{j/2} \varphi(2^j t - k)$  is called the system of Haar scaling functions. For each  $j, k \in \mathbb{Z}$ , the collection of  $\{\varphi_{i,k}(t)\}_{i,k \in \mathbb{Z}}$  is called the Haar scaling function at scale  $j$ .

Haar scaling function can be defined as

$$\varphi(t) = \chi_{[0,1)}(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

For each  $j, k \in \mathbb{Z}$ ,  $\{\varphi_{i,k}(t)\}_{i,k \in \mathbb{Z}} = 2^{j/2} \varphi(2^j t - k) = D_{2^j} T_k \varphi(t)$ , where dilation operator  $D_a f(x) = a^{-1/2} f(ax)$  and Translation of operator  $T_k f(x) = f(x-k)$ .

**Haar wavelet system: Definition (Haar wavelet system):** For each  $j, k \in \mathbb{Z}$ , we define The family of functions  $\{\psi_{i,k}(t)\}_{i,k \in \mathbb{Z}}$  is called the Haar wavelet system.

Consider  $f(t)$  is defined on  $L_2[0,1]$ , has an expansion in terms of Haar functions as follows: For any integer  $J \geq 0$ ,

$$\begin{aligned} f(t) &= \sum_{k=0}^{2^J-1} \langle f, \varphi_{J,k} \rangle \varphi_{J,k}(t) + \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \\ &= \sum_{k=0}^{2^J-1} c_{J,k} \varphi_{J,k}(t) + \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t) \end{aligned} \quad (8)$$

which is known as Haar series and  $d_{j,k}$  &  $c_{j,k}$  are the Haar coefficients for wavelet and Haar scaling coefficients respectively. For details see Walnut (2001).

## Results

### Approximation by Haar wavelet in different smoothness spaces

**Approximation space:** Let  $(X, \|\cdot\|_X)$  be a normed space in which the approximation takes place. Let  $(S_N)_{N \geq 0}$  be a family of sub-spaces of a normed space  $X$ . Our approximation comes from the space  $(S_N)_{N \geq 0} \subset X$ .

For a function  $f \in X$ , the approximation error is  $E_N(f)_X = \text{dist.}(f, S_N)_X = \inf_{g \in S_N} \|f - g\|_X$

where  $g$  is the approximating function in  $(S_N)_{N \geq 0}$  and  $N$  is dimension of  $S_N$ . In most cases of interest,  $E_N(f)$  goes to zero as  $N$  tends to infinity.

**Approximation in  $L_2(\mathfrak{R})$ :** Let  $f$  be continuous on  $L_2(\mathfrak{R})$  and the Haar wavelet series of  $f$  is

$$f \approx \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) = \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t). \quad (10)$$

If  $\psi_{j,k}(t)$  is supported on the interval  $I_{j,k} = [2^j k, 2^j(k+1)]$ , then the wavelet coefficients

$$d_{j,k} = \langle f, \psi_{j,k} \rangle = \int_R f(t) \psi_{j,k}(t) dt = 2^{-j/2} \int_{k2^j}^{(k+1)2^j} f(t) \psi(2^j t - k) dt \quad (11)$$

For computing finite sum, let  $N = 2^J$  be coefficients for some  $J \in \mathbb{N}$ .

i.e. we consider  $j = 0, 1, 2, 3, \dots, J-1$ , then  $\sum_{j=0}^{J-1} \sum_{n=0}^{2^j-1} 1 = 1 + 2 + 2^2 + \dots + 2^{J-1} = N - 1$  coefficients.

For Haar wavelet we can see that for each  $j$  only one of the coefficients is non-zero and its size is  $d_{j,k} \approx 2^{-j/2}$ . For details one can see Christensen and Christensen (2004) and Walnut (2001).

Therefore the error of the Haar approximation in  $L_2(\mathfrak{R})$  is

$$\begin{aligned} & \left\| f - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_2}^2 \\ &= \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_2}^2 \\ &= \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \left| \langle f, \psi_{j,k} \rangle \right|^2 \approx \sum_{j=J}^{\infty} 2^{-j} \approx 2^{-J} \approx \frac{1}{N} = O(2^{-J}) \end{aligned} \quad (12)$$

**Approximation in  $L_p(\mathfrak{R})$  : Theorem:** If  $f \in L_p(R)$  and the partial sum of the Haar wavelet series of  $f$  is

$g = \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$ , for  $j \in N$ . Then the error of the approximation is  $O(2^{-J/2})$ .

**Proof:** The error of the approximation in  $L_p(\mathfrak{R})$  is

$$\begin{aligned} \|f - g\|_{L_p} &= \left\| f - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_p} \\ &= \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_p} \\ &\approx \left( \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^p \right)^{1/p} \approx \left( \sum_{j=J}^{\infty} 2^{-\frac{jP}{2}} \right)^{1/p} \approx 2^{-\frac{J}{2}} = O(2^{-\frac{J}{2}}). \end{aligned} \quad (13)$$

**Approximation in  $Lip_M(\alpha, L_p)$  spaces : Theorem:**

If  $f \in Lip_M(\alpha, L_p)$ ,  $0 < \alpha \leq 1$ ,  $1 < p \leq \infty$ ,  $M > 0$ , and  $g(t) = \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$

is the Haar wavelet series of  $f$  for some  $J \in N$ , then the error of the approximation in  $Lip_M(\alpha, L_p)$  is  $O(2^{-J\alpha})$ .

**Proof:** From de Vore (1998) we have if  $f \in Lip_M(\alpha, L_p)$ ,  $0 < \alpha \leq 1$ ,  $1 < p \leq \infty$ ,

$$dist(f, S_N)_p \leq \inf_{g \in S_N} \|f - g\|_p \leq C_p \|f\|_{Lip(\alpha, L_p)} \delta^\alpha, \text{ where } \delta = \max_{0 \leq k < N} |t_{k+1} - t_k|$$

So the error of the approximation in  $Lip_M(\alpha, L_p)$  is

$$\begin{aligned} \|f - g\|_{L_p} &= \left\| f - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_p} \\ &\leq C_p \|f\|_{Lip(\alpha, L_p)} (2^{-J})^\alpha \\ &\leq M (2^{-J})^\alpha = O(2^{-J\alpha}), \end{aligned} \quad (14)$$

where  $C_p$  depending on  $p$ .

**Approximation in Sobolev spaces  $H^m(\mathfrak{R})$ : Theorem:** If

$f \in H^m(\mathfrak{R})$  and  $g(t) = \sum_{j=0}^{J-1} \sum_{k=0}^N \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$  is the finite Haar wavelet series of  $f$  for some

$J \in \mathbb{N}$ , then the error of the approximation is  $O\left(2^{-\frac{mN}{2}}\right)$ , where  $N = 2^J$ .

**Proof:** The error of the approximation is

$$\begin{aligned} \|f - g\|_{L_2} &= \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_2} \\ &\leq \left( \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} \frac{2^{mk}}{2^{mN}} |\langle f, \psi_{j,k} \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \left( 2^{-mN} \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{mk} |\langle f, \psi_{j,k} \rangle|^2 \right)^{\frac{1}{2}} \right) \end{aligned}$$

By using the properties of Besov space we have  $\|f\|_{H^m(L_2(\mathfrak{R}))} \cong \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{mk} |\langle f, \psi_{j,k} \rangle|^2 \right)^{\frac{1}{2}}$ .

$$\text{Therefore } \|f - g\|_{L_2} \leq 2^{-\frac{mN}{2}} \|f\|_{H^m(L_2(\mathfrak{R}))} = O\left(2^{-\frac{mN}{2}}\right). \quad (15)$$

**Approximation in Besov space  $B_q^\alpha(L_p(\mathfrak{R}))$ : Theorem:**

If  $f \in B_q^\alpha(L_q(\mathfrak{R}))$ ,  $\alpha > 0$ ,  $0 < q \leq \infty$  and  $g(t) = \sum_{j=0}^{J-1} \sum_{k=0}^N \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$  is the finite

Haar wavelet series of  $f$  for some  $J \in \mathbb{N}$ , then the error of the approximation is

$$O\left(2^{-\frac{\alpha N}{2}}\right), \text{ where } N = 2^J.$$

**Proof:** The error of the approximation is

$$\begin{aligned} \|f - g\|_{L_q} &= \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_q} \\ &\leq \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} |\langle f, \psi_{j,k} \rangle|^q \right)^{\frac{1}{q}} \leq \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} \frac{2^{\alpha k}}{2^{\alpha N}} |\langle f, \psi_{j,k} \rangle|^q \right)^{\frac{1}{q}} \\ &\leq \left( 2^{-\frac{\alpha N}{q}} \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{\alpha k} |\langle f, \psi_{j,k} \rangle|^q \right)^{\frac{1}{q}} \right) \end{aligned}$$

By using the properties of Besov space we have  $\|f\|_{B_q^{\alpha}(L_q(\mathfrak{R}))} \cong \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{\alpha k} |\langle f, \psi_{j,k} \rangle|^q \right)^{\frac{1}{q}}$ .

Therefore  $\|f - g\|_{L_q(\mathfrak{R})} \leq 2^{-\frac{\alpha N}{q}} \|f\|_{B_q^{\alpha}(L_q(\mathfrak{R}))} = O(2^{-\frac{\alpha N}{q}})$ , where  $\frac{1}{q} = \frac{\alpha}{2} + \frac{1}{2}$ . (16)

### Discussion

In approximation theory our main goal is how we improve our approximation order. i.e. how we get better approximation of our target function. Approximation by wavelet is the new dimension in the approximation theory. In our study, we approximate any general functions in  $L_2(\mathfrak{R})$  by Haar wavelet in different smoothness spaces such as Lebesgue space, Lipschitz continuous space, Sobolev and Besov spaces. We have seen that error is decreasing if the smoothness of spaces is improved. i.e. we are getting better approximation if the smoothness of spaces is improved. One can use this idea to get better approximation of their target function such as in image processing, data storage, signal processing, Medical technology etc.

### Conclusion

The above theorems show that the approximation order will improve if the smoothness of the approximation spaces is improved.

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