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# INVARIANCE: AN ANALOGY BETWEEN SYMMETRY AND CONSERVATION LAWS

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**Abstract**: Invariance plays an essential role in Physics and Mathematics. It has a close connection between symmetry and conserved quantity. In general any symmetry of the Lagrangian as well as Hamiltonian corresponds to a conserved quantity, and vice versa. This elegant works were first discussed and formalized by Emmy Neother. In this study an analogy is shown between invariance, symmetry and conservation laws.

**Key words**: Invariance, symmetry, conserved quantity, Lagrangian and Hamiltonian systems, Neother theorem.

## Introduction

The study of invariance problems was pioneered in the early part of 20th century by Noether (1918), influenced by the work of Klein and of Lie on the transformation properties of differential equations under continuous groups of transformations. Lewis first introduced the concept of invariants for time dependent oscillator system, which has attracted considerable interest in the Literature in both Classical and Quantum mechanics (Ryder, 1985; Synge and Griffith, 1959). Invariance applies in the sense of constantness or conservation, when some characteristics of the motion of a system remain constant in time. If a physical system remains unaffected or unchanged with respect to some disturbances (i.e. transformations), the system is called invariant under those transformations (Gupta *et al.*, 1994). In our physical word, there exist a number of conservation principles or laws, some are exact and some are approximate. There are conservation laws relating to energy, linear momentum, angular momentum, charge and various other quantities. They have been established as a result of extensive research on particle systems. Invariance is an important and a very powerful tool in solving mechanical problems in many respects of Mathematics. When a physical system remains invariant, one can extract a great deal of information about the physical systems. It tells us a great deal about the motion even if forces affecting motion are not known in advance. Even when the force is known exactly, the invariance may provide a convenient method of solving the motion of the particle and particle systems. Another important technique, Calculus of variation (Weinstocks, 1952; Gelfand and Fomin, 1963; Das, 1998) was used to generalize Euler-Lagrange equation (Biswas, 2005) which is also invariant under canonical transformations.

## Symmetry

Symmetry is a fundamental attribute of the natural world that enables an investigator to study particular aspects of physical system by themselves. Various types of symmetries occur frequently

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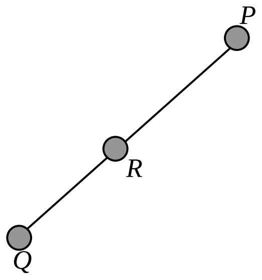
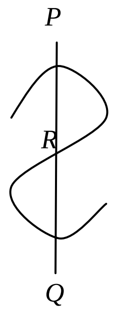
in nature and in mathematics. Recognizing the symmetry of a mathematical object is important because it simplifies the study of the subject. There are two types of symmetries: dynamical and geometrical. There is more to be said about symmetry of the laws of nature and that concerns conservation laws. For example, the assumption that the space is homogenous or possesses traditional symmetry, leads to the conclusion that the linear momentum of a closed isolated system does not change as the system moves. This makes it possible to study separately the motion of the center of mass and the inertial motion of the system. In similar fashion, the assumption that the space is isotropic or possesses rotational symmetry means that the total angular momentum of such a system is conserved. Thus Symmetry in either system relates to conservation theorems (Goldstein, 1950; Gupta, 1997). Let us mention the symmetries of the laws of nature that are related to the conservation referred to above and show how each conservation can be derived analytically from its related symmetry for a simple mechanical system.

### Symmetry in Geometry

We have already mentioned that various types of symmetries occur frequently in nature and in mathematics. From the very first beginning we have tried to realize the symmetry in geometrical configuration.

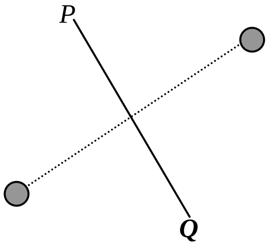
#### Symmetry about a point





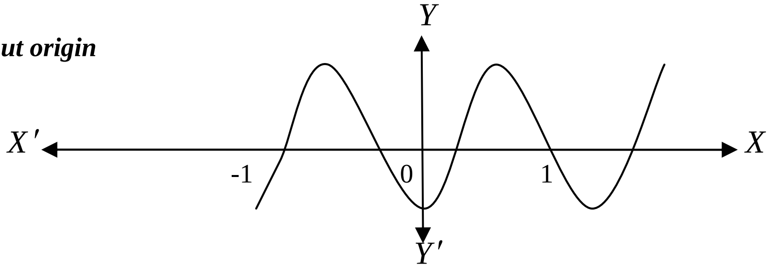
Two points *P* and *Q* are symmetric about a third point *R*, if *R* is the midpoint of the line segment *PQ*. A curve is symmetric about the point R, if for any point P on the curve, the point Q which is symmetric to P about R is also on the curve.

#### *Symmetry about a line*



Two points *P* and *Q* are symmetric about a line if the line is perpendicular bisector of the line segment *PQ*.

#### *Symmetry about origin*



The graph of an equation is symmetric about the origin if replacing x by (-x) and y by
(-y) in the equation gives an equivalent equation.

*Example:* The curve $y = \sin x$ is symmetric about the origin.

**Symmetry in General**

In general the symmetry properties of any things may be appeared under a system. The system may be abstract or concrete, microscopic or macroscopic, static or dynamic, etc.

A transformation of a system is a mapping of a state space of the system into itself. We can denote a transformation *T* by

$$u \xrightarrow{T} v \quad \text{or,} \quad v = T(u)$$

Where $u$ and $v$ represent states of the system.

The transformations that map every state to an image state equivalent to the object state for a state space with equivalence relation, such transformations are called a symmetry transformation. Thus a symmetry transformation *S* is defined as

$$u \xrightarrow{s} v \equiv u$$

$$\text{or,} \quad S(u) \equiv v = u$$

for all states $u$.

The set of all invertible symmetry transformations of a state space of system for an equivalence relation forms a group, a subgroup of the transformation group, called the symmetry group of the systems for the equivalence relation:

- Closure law
- Associative law
- Existence of identity
- Inverse law

**Symmetry in Classical Mechanics**

There is more to be said about symmetry of the laws of nature and that concerns conservation laws, such as conservation of energy, conservation of linear momentum, conservation of angular momentum.

Let us mention the symmetries of the laws of nature that are related to the conservation referred to above and show how each conservation can be derived analytically from its related symmetry for a simple mechanical system.

Let us consider the non-relativistic system of a single point particle of mass $m$ moving in one dimension in a potential $V$ that is function of the particle's coordinate $x$.

Then its total energy is

$$E = \frac{1}{2} m\dot{x}^2 + V \qquad \left[ \dot{x} = \frac{dx}{dt} = v = \text{Velocity} \right]$$

$$\therefore \frac{dE}{dt} = m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x}$$

The derivative $\frac{dV}{dx}$ is, by the definition of potential, the negative of the force on the particle, which in turn; by Newton's second law, equals $\left(-m\ddot{x}\right)$, giving

$$\frac{dE}{dt} = m\dot{x}\ddot{x} - m\dot{x}\ddot{x} = 0$$

Which implies that the total energy does not change with time; it is conserved.

## Conservation Laws

### Linear Momentum Conserved

We have been concerned primarily with obtaining the equations of motion, and little has been said about how to solve them for a particular problem once they have been obtained.

In many problems a number of first integrals of the equations of motion can be obtained immediately; by this we mean relations of the type

$$f\left(q_1, q_2, ......, \dot{q}_1, \dot{q}_1, .........., t\right) = \text{constant} \tag{3.1.01}$$

which are first order differentials equations. These first integrals are of interest because they tell us something physically about the system. They include, in fact, the conservation laws. Now we will show the momentum conservation.

We know the Lagrangian of a closed system is translationally invariant. Let the Lagrangian be $L \equiv L(q_i, \dot{q}_i) = L(r_i, v_i)$. Consider an infinitesimal parallel displacement $\in$ of the system. Every particle of the system is shifted by the same amount, the position vector $r_i$ becoming $r_i + \in$. Thus the change in the Lagrangian, the velocities remaining fixed, is given by

$$\delta L = \sum_i \frac{\partial L}{\partial r_i} . \delta r_i = \in . \sum_i \frac{\partial L}{\partial r_i} \tag{3.1.02}$$

Then, the condition $\delta L = 0$ is equivalent to the requirement $\sum_i \frac{\partial L}{\partial r_i} = 0,$ since is arbitrary.

From Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = \frac{\partial L}{\partial r_i},$$

We obtain that the translational invariance of the Lagrangian leads to the condition

$$\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = 0,$$

$$\frac{d}{dt} \sum_i \frac{\partial L}{\partial v_i} = 0, \tag{3.1.03}$$

or $$\frac{d}{dt} \mathrm{P} = 0,$$

so the total linear momentum P is a constant or an integral of motion.

Conversely, if this conservation law is to be an automatic consequence of the equations of motion for all states of motion, the corresponding $L$ must be invariant under translations of the coordinate axes. Clearly linear momentum is an additive quantity.

If the Lagrangian of a system does not contain a given coordinate $q_j$, then the coordinate is said to be *cyclic* or *ignorable*. It is the customary one and will be used here. The Lagrange equation of motion,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_j} = 0$$ reduces for a cyclic coordinate, to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$$

or

$$\frac{dp_j}{dt} = 0,$$

which means that

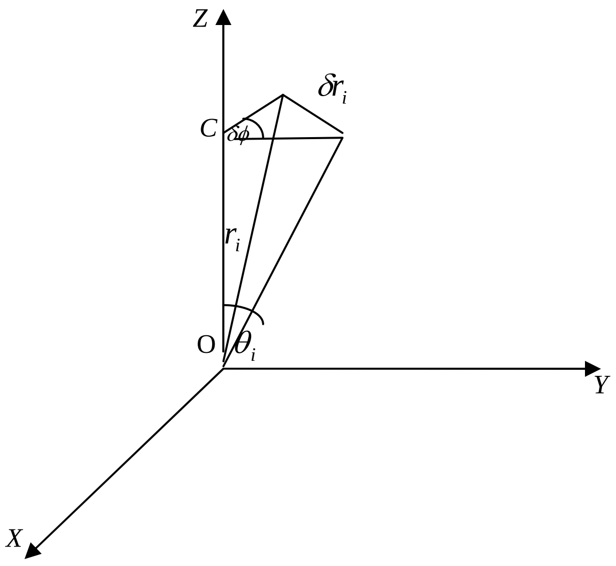
$$p_j = \text{constant} \tag{3.1.04}$$

Hence we can state as a general conservation theorem that the generalized momentum conjugate to a cyclic coordinate is conserved. Equation (3.1.03) constitutes a first integral of the form (3.1.01) for the equations of motion.

So in short, in a system consisting of interacting particles, not subject to an external force, the total momentum is always conserved.

**Angular Momentum Conserved**

Now consider the isotropy of space. This property tells us that the mechanics of a closed system should not change when the system is rotated as a whole, in any manner, in ordinary space.



Let $r_i$ denote the radius vector for the $i$ th particle of the system, O being the origin of the Cartesian frame of reference. Also let the system as a whole undergo an infinitesimal rotation $\delta\phi$ .Corresponding to this rotation, the position of the $i$ th particle is shifted from $P$ to $P'$ and the radius $r_i$ is increased by an amount $\delta r_i$ given by

$$\delta r_i = \delta\phi \times r_i \tag{3.2.01}$$

$$|\delta r_i| = r_i \sin\theta_i \, \delta\phi$$

when the system is rotated, not only the position vectors but also the velocity vectors of the particles change their direction, and all the vectors get transformed in the same manner. Hence, the corresponding change in the velocity vector $v_i$ of the $i$ th particle is given by

$$\delta v_i = \delta\phi \times v_i$$

Now $$L \equiv L(q_i, \dot{q}_i) = L(r_i, v_i)$$

Hence, the condition that the Lagrangian is rotationally invariant is converted into condition

$$\begin{aligned} 0 = \delta L &= \sum_i \left[ \frac{\partial L}{\partial r_i} . \delta r_i + \frac{\partial L}{\partial v_i} . \delta v_i \right] \\ &= \sum_i \left[ \dot{p}_i . \delta r_i + p_i . \delta v_i \right] \\ &= \sum_i \left[ \dot{p}_i . (\delta\phi \times r_i) + p_i . (\delta\phi \times v_i) \right] \\ &= \delta\phi \sum_i \left[ (r_i \times \dot{p}_i) + (v_i \times p_i) \right] \\ &= \delta\phi . \frac{d}{dt} \sum_i (r_i \times p_i) = \delta\phi . \frac{dM}{dt} \end{aligned}$$

Since the above is true for all arbitrary values of $\delta\phi$, it follows that the condition reduces to

$$\dot{M} = 0$$

or, $M =$ constant

We see that the rotational invariance of the Lagrangian of a closed system is equivalent to the conservation theorem of total angular momentum $M$ of the system. $M$ is also an additive quantity. Conversely, if the conservation of $M$ is to follow automatically from the equations of motion, $L$ must be invariant under rotations of the Cartesian axes.

**Total Energy Conserved**

If any system or any function representing a property of the system remains invariant under some operations then the system is said to possess symmetry with respect to the given operation. They have been established as a result of extensive research on particle systems.

We consider a conservative system such that the forces acting on the system are derivable from a potential, $F = -\nabla U$ . We know that the Lagrangian of a closed system cannot be an explicit function of time *t*. In other words, we presume that the constraints are independent of time. Thus we write

$$L = L(q_i, \dot{q}_i),$$

so that the total time derivative of *L* is

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i} \frac{dq_i}{dt} + \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \qquad (3.3.01)$$

Using Lagrange’s equation, Eq. (3.3.01) becomes

$$\frac{dL}{dt} = \sum_i \dot{q}_i \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_i} \right) + \sum_i \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$= \sum_i \frac{d}{dt}\left( \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right),$$

or
$$\frac{d}{dt}\left( \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right) = 0 \qquad (3.3.02)$$

Defining the quantity in the bracket in Eq. (3.3.02) as *E*, we see that this is integral of motion:

$$E \equiv \left( \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right) = \text{a constant} \qquad (3.3.03)$$

we shall now show that *E*, thus defined is just the total energy of the system. Now

$$L = T - U$$

and for conservative systems, both *T* and *L* are homogenous functions of the $\dot{q}_i\text{'}s$. Now, the Euler formula $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i = mf$ holds for a homogenous function $f(x_1,........., x_N)$ of the *m*th degree. Applying this formula to *T*, which is a homogenous function of the velocities of second degree, we obtain

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T \qquad (3.3.04)$$

Again, for conservative systems, the potential *U* is independent of velocities, so that

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} \qquad (3.3.05)$$

Thus we conclude

$$E = \sum_i \dot{q}_i p_i - L = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - L = 2T - (T - U) = T + U \qquad (3.3.06)$$

which is the total energy of the system.

The law of conservation of energy is valid not only for closed systems, but also for those, which are in a constant external field since for such systems too; *L* has no explicit time dependence.

## Discussion

It is believed that the space around us is homogeneous and isotropic. In fact, if we close ourselves in a room in a particular place it is extremely difficult to know whether we are in Bangladesh or any other place (Biswas, 1998). We really do not know where we are. A particular physical observation is made at one place will not be different at another place. That is physical result is unchanged by space translation. From the above discussion of the context of situation we find that if a Lagrangian for N particle system be invariant (Pinch, 1993) under infinitesimal translation then the total linear moment is conserved. If in analogy with space translation we assume the Lagrangian to be invariant under time-translation, then the total energy is conserved. Also the rotational invariance of the Lagrangian implies the conservation of angular momentum. That is rotational symmetry means conservation of angular momentum. In the similar fashion, it can be shown that if the Hamiltonian H is invariant under time-translation, then total energy is conserved. Likewise, if H is invariant under infinitesimal space-translation, then total linear momentum is conserved. Lastly, if Hamiltonian is invariant under 3-dimensional Euclidean rotation, then angular momentum is conserved.

## Conclusion

Finally, we have made a tremendous conclusion that whenever a system either Lagrangian or Hamiltonian is invariant under a suitable transformation, the corresponding symmetries are obtained which are called the conservation laws. That is due to symmetry the space around is completely isotropic and homogenous. So, invariance is nothing but an analogy between symmetry and conserved quantity.

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