



INVESTIGATION OF THE SOLUTION OF A FREDHOLM INTEGRAL EQUATION OF FIRST KIND WITH RESPECT TO THE FOURIER SERIES

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Abstract: When an integral equation is solved by using the Fourier series then the solution represents a stationary signal. Usually, integral equation is solved by the successive approximation method and the resolvent kernel method in which the solution is not of Fourier series type and this solution does not represent a stationary signal. In this paper, our main goal is to determine the solution of a Fredholm integral equation of first kind by using the Fourier series.

Key words: Fourier series, Fredholm integral equation, kernel, stationary signal

Introduction

Integral equations are one of the most useful techniques in many branches of pure and applied mathematics, particularly in boundary value problems in the ordinary and partial differential equations. Integral equations occur in many fields of mechanics and mathematical physics. They are also related to the problems in mechanical vibrations, theory of analytic functions, orthogonal systems, and quadratic forms of infinitely many variables. Integral equations arise in several problems of science and technology and may be obtained directly from physical problems, i.e. radiation transfer problem and neutron diffusion problem etc. They also arise as representation formula for the solutions of differential equations; a differential equation can be replaced by an integral equation with the help of initial and boundary conditions. As such, each solution of the integral equation automatically satisfies these boundary conditions (Swarup, 1996; Jeffry, 1976; Tricomi, 1957; Wirda, 1930).

The actual development of the theory of integral equations began only at the end of 19th century due to the works of Italian mathematician V. Volterra and principally to the year 1900, in which the Swedish mathematician I. Fredholm published his work on the method of solution for the Dirichlet problem (Williams, 1978; Peters, 1972; Pogorzelski, 1966; Mikhlin, 1957). In this paper, we have included a new technique for determining the solution of a Fredholm integral equation of first kind whose name is Fourier series method.

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Materials and Method

In this study, we have used Fourier series method for determining the solution of a Fredholm integral equation of first kind and also used MATLAB for drawing the graph of the solution.

Integral Equation: An integral equation is an equation in which an unknown function to be determined appears under one or more integral signs.

Examples: 1. $\phi(x) = F(x) + \lambda \int_a^b k(x,t)\phi(t)dt$

2. $\phi(x) = \lambda \int_a^b k(x,t)\phi(t)dt$

The function $\phi(x)$ in (1) and (2) is the unknown function to be determined, while all other functions are known. The function $k(x,t)$ is called the kernel of the integral equation and λ is a non-zero real or complex parameter.

If Ω is the domain of the variable t , we often write an integral equation as

$$\phi(x) = F(x) + \lambda \int_{\Omega} k(x,t)\phi(t)dt$$

Fredholm integral equation: An integral equation is said to be a Fredholm integral equation, if the domain of integration is fixed.

Example: $\alpha(x)\phi(x) = F(x) + \lambda \int_a^b k(x,t)\phi(t)dt$ (1)

- i. If $\alpha(x) = 0$, then equation (1) reduces to $F(x) = \lambda \int_a^b k(x,t)\phi(t)dt$, this equation is said to be a Fredholm integral equation of first kind. The functions $k(x,t)$ and $F(x)$ and the limits a and b are known. It is proposed to determine the unknown function $\phi(x)$ so that (1) is satisfied for all values of x in the closed interval $a \leq x \leq b$ and $k(x,t)$ is the kernel of this equation.
- ii. If $\alpha(x) = 1$, then equation (1) reduces to $\phi(x) = F(x) + \lambda \int_a^b k(x,t)\phi(t)dt$, this equation is said to be a Fredholm integral equation of second kind.
- iii. If $F(x) = 0$, $\alpha(x) = 1$, then equation (1) reduces to $\phi(x) = \lambda \int_a^b k(x,t)\phi(t)dt$, this equation is said to be a homogeneous Fredholm integral equation of second kind.

Solution of Fredholm integral equation of first kind:

Given an integral equation

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-\alpha^2}{1-2\alpha \cos(x-y) + \alpha^2} \phi(y) dy, \quad \text{where } 0 < \alpha < 1, \quad -\pi \leq x \leq \pi \quad (2)$$

which is of the form, $f(x) = \lambda \int_{-\pi}^{\pi} k(x,y)\phi(y)dy$ i.e. Fredholm integral equation of first kind

$$\text{Now, } \frac{1-\alpha^2}{1-2\alpha \cos(x-y) + \alpha^2} = -1 + \frac{-\alpha(e^{iz} + e^{-iz}) + 2}{(1-\alpha e^{iz})(1-\alpha e^{-iz})} \quad (3)$$

$$\text{Let, } \frac{-\alpha(e^{iz} + e^{-iz}) + 2}{(1-\alpha e^{iz})(1-\alpha e^{-iz})} = \frac{A}{(1-\alpha e^{iz})} + \frac{B}{(1-\alpha e^{-iz})} \quad (4)$$

Solving equation (4), we get

$$A = 1, B = 1$$

Putting the values of A and B in (4), we get

$$\frac{-\alpha(e^{iz} + e^{-iz}) + 2}{(1 - \alpha e^{iz})(1 - \alpha e^{-iz})} = \frac{1}{(1 - \alpha e^{iz})} + \frac{1}{(1 - \alpha e^{-iz})} \quad (5)$$

Using the equation (5) in (3), we obtain

$$\frac{1 - \alpha^2}{1 - 2\alpha \cos(x - y) + \alpha^2} = 1 + 2 \sum_{n=1}^{\infty} \alpha^n (\cos nx \cos ny + \sin nx \sin ny) \quad (6)$$

The above series is absolutely convergent.

We know that the Fourier series for $f(x)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (7)$$

$$\text{Let } \phi(y) = \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos ny + d_n \sin ny) \quad (8)$$

be the solution of the integral equation (2).

Using the equation (6), (7) and (8) in (2), we obtain

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\left\{ 1 + 2 \sum_{n=1}^{\infty} \alpha^n (\cos nx \cos ny + \sin nx \sin ny) \right\} \times \right. \\ &\quad \left. \left\{ \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos ny + d_n \sin ny) \right\} \right] dy \\ &= \int_{-\pi}^{\pi} \left[\left\{ \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n (\cos nx \cos ny + \sin nx \sin ny) \right\} \times \right. \\ &\quad \left. \left\{ \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos ny + d_n \sin ny) \right\} \right] dy \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi} \cdot \frac{c_0}{2} dy + \int_{-\pi}^{\pi} \frac{c_0}{2\pi} \sum_{n=1}^{\infty} \alpha^n \cos nx \cos ny dy + \int_{-\pi}^{\pi} \frac{c_0}{2\pi} \sum_{n=1}^{\infty} \alpha^n \sin nx \sin ny dy \\ &\quad + \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{n=1}^{\infty} c_n \cos ny dy + \int_{-\pi}^{\pi} \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n c_n \cos nx \cos^2 ny dy + \\ &\quad \int_{-\pi}^{\pi} \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n c_n \sin nx \sin ny \cos ny dy + \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{n=1}^{\infty} d_n \sin ny dy + \\ &\quad \int_{-\pi}^{\pi} \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n d_n \cos nx \sin ny \cos ny dy + \int_{-\pi}^{\pi} \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n d_n \sin nx \sin^2 ny dy \end{aligned}$$

$$\begin{aligned}
 &= \frac{c_0}{4\pi} \int_{-\pi}^{\pi} dy + \frac{c_0}{2\pi} \sum_{n=1}^{\infty} \alpha^n \cos nx \int_{-\pi}^{\pi} \cos ny dy + \frac{c_0}{2\pi} \sum_{n=1}^{\infty} \alpha^n \sin nx \int_{-\pi}^{\pi} \sin ny dy \\
 &\quad + \frac{1}{2\pi} \sum_{n=1}^{\infty} c_n \int_{-\pi}^{\pi} \cos ny dy + \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n c_n \cos nx \int_{-\pi}^{\pi} \cos^2 ny dy + \\
 &\quad \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n c_n \sin nx \int_{-\pi}^{\pi} \sin ny \cos ny dy + \frac{1}{2\pi} \sum_{n=1}^{\infty} d_n \int_{-\pi}^{\pi} \sin ny dy + \\
 &\quad \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n d_n \cos nx \int_{-\pi}^{\pi} \sin ny \cos ny dy + \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n d_n \sin nx \int_{-\pi}^{\pi} \sin^2 ny dy \\
 &= \frac{c_0}{4\pi} [y]_{-\pi}^{\pi} + 0 + 0 + 0 + \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n c_n \cos nx \cdot \pi + \frac{1}{2\pi} \sum_{n=1}^{\infty} \alpha^n c_n \sin nx \int_{-\pi}^{\pi} \sin 2ny dy + 0 \\
 &\quad + \frac{1}{2\pi} \sum_{n=1}^{\infty} \alpha^n d_n \cos nx \int_{-\pi}^{\pi} \sin 2ny dy + \frac{1}{\pi} \sum_{n=1}^{\infty} \alpha^n d_n \sin nx \cdot \pi
 \end{aligned}$$

$$[\because \int_{-\pi}^{\pi} \cos nx dx = \int_{-\pi}^{\pi} \sin nx dx = 0 \text{ and } \int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \cos^2 nx dx = \pi]$$

$$\begin{aligned}
 \text{or } \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) &= \frac{c_0}{4\pi} \cdot 2\pi + \alpha^n \sum_{n=1}^{\infty} c_n \cos nx + 0 + 0 + \alpha^n \sum_{n=1}^{\infty} d_n \sin nx \\
 &= \frac{c_0}{2} + \alpha^n \sum_{n=1}^{\infty} (c_n \cos nx + d_n \sin nx)
 \end{aligned}$$

$$\therefore \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{c_0}{2} + \alpha^n \sum_{n=1}^{\infty} (c_n \cos nx + d_n \sin nx) \quad (9)$$

Equating the like terms of equation (9), we get

$$c_0 = a_0, c_n = \alpha^{-n} a_n \text{ and } d_n = \alpha^{-n} b_n \quad (10)$$

Using the equation (10) in (8), we get

$$\phi(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\alpha^{-n} a_n \cos ny + \alpha^{-n} b_n \sin ny) \quad (11)$$

which is the required solution of the given integral equation.

Results

The solution of the Fredholm integral equation of first kind solving by the Fourier series is also of Fourier series type and this solution represents a stationary signal (a stationary signal is a signal where there is no change of the properties of signal) which is shown in the Fig. 1, Fig. 2 and Fig. 3 clearly.

Discussion

In this section the solution of the Fredholm integral equation of first kind are presented graphically. We have drawn Fig. 1, Fig. 2 and Fig. 3 for the solution of the Fredholm integral equation of first kind by using MATLAB for definite terms of the series and increasing the values of the domain by different step sizes in the interval $[-\pi, \pi]$.

Fig.1 is drawn for $a_0 = .1, \alpha = .8, a_n = .3, b_n = .2, n = 10$ and step size $y = .01$ in $[-\pi, \pi]$.

Fig. 2 is drawn for $a_0 = .1, \alpha = .8, a_n = .3, b_n = .2, n = 10$ and step size $y = .5$ in $[-\pi, \pi]$ and

Fig. 3 is drawn for $a_0 = .1, \alpha = .8, a_n = .3, b_n = .2, n = 20$ and step size $y = .01$ in $[-\pi, \pi]$. From the Fig. 1 and Fig. 2, we observe that when the step size y is increasing and the values of a_0, α, a_n, b_n and n is keeping fixed, then the wave length is increasing but frequency number is decreasing. Again, from the Fig. 1 and Fig. 3, we observe that when n is increasing and all other values are keeping fixed i.e. the values of a_0, α, a_n, b_n and step size y are unchanged, then the frequency number is increasing but the wave length is decreasing.

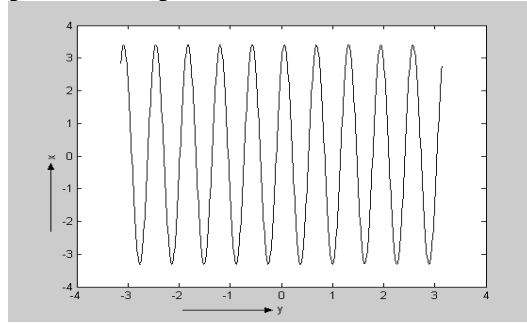


Fig.1. Graph of (11) for $a_0 = .1, \alpha = .8, a_n = .3, b_n = .2, n = 10$ and $y = .01$

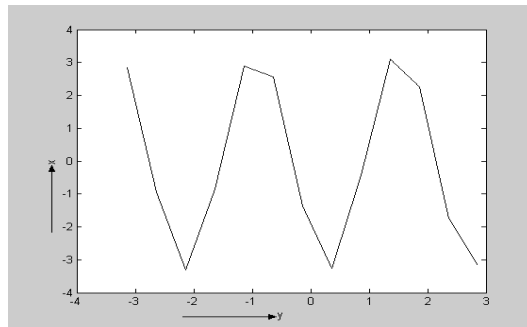


Fig.2. Graph of (11) for $a_0 = .1, \alpha = .8, a_n = .3, b_n = .2, n = 10$ and $y = .5$

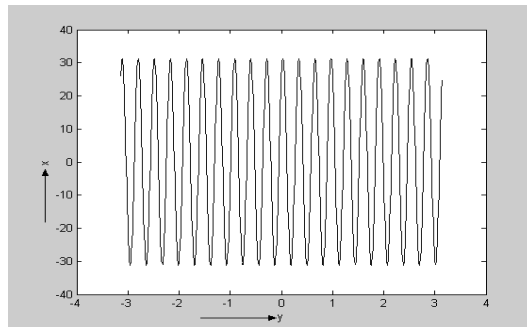


Fig.3. Graph of (11) for $a_0 = .1, \alpha = .8, a_n = .3, b_n = .2, n = 20$ and $y = .01$

Conclusion

We know that Fourier series is a powerful tool for analyzing the stationary signal and periodic function is one kind of stationary signal. Also we know that *sine* and *cosine* functions are the examples of the periodic functions. Since *sine and cosine* terms are present in the solution of the Fredholm integral equation of first kind, hence from our discussion, we can say that the solution of the Fredholm integral equation of first kind is also a stationary signal. Finally, we conclude that Fourier series is a very interesting, powerful and easy technique for determining the solution of the Fredholm integral equation of first kind.

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