



**ANALYTICAL SOLUTION OF THE HILBERT SINGULAR INTEGRAL EQUATION WITH RESPECT TO THE HILBERT FORMULA**

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**Abstract:** The objective of this paper is to solve the Hilbert singular integral equation of first and second kind by using the Hilbert formula and some known results are obtained as special cases. Applications of these singular integral equations to Boundary value problems of Elasticity and allied subjects are well known.

**Key words:** Fredholm integral equation, Hilbert formula, kernel

**Introduction**

Integral equations are one of the most useful techniques in many branches of pure and applied mathematics, particularly on account of its importance in boundary value problems in the ordinary and partial differential equations. Integral equations occur in many fields of mechanics and mathematical physics. They are also related with the problems in mechanical vibrations, theory of analytic functions, orthogonal systems, and quadratic forms of infinitely many variables. Integral equations arise in several problems of science and technology and may be obtained directly from physical problems, i.e. radiation transfer problem and neutron diffusion problem etc. They also arise as representation formula for the solutions of differential equations; a differential equation can be replaced by an integral equation with the help of initial and boundary conditions. Such as, each solution of the integral equation automatically satisfies these boundary conditions (Jeffry (1976); Swarup (1996); Tricomi (1957); Wirda (1930); Williams (1978)).

The name 'Integral Equation' for any equation involving the unknown function  $\varphi(x)$  under the integral sign was introduced by Bois-reymond in 1888. In 1782, Laplace used the integral transform

$f(x) = \int_0^{\infty} e^{-xt} f(t) dt$  to solve the linear integral equations and differential equations. In 1826, Abel

solved the singular integral equation named after him having the form

$$f(x) = \int_0^x \frac{\varphi(t)}{(x-t)^\alpha} dt,$$

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where  $f(x)$  is a continuous function satisfying  $f(a) = 0$  and  $0 < \alpha < 1$ . Huygens solved the Abel's integral equation for  $\alpha = 1/2$ . In 1826, Poisson obtained an integral equation of the type

$$\varphi(x) = f(x) + \lambda \int_0^x k(x,t) \varphi(t) dt$$

in which the unknown function  $\varphi(t)$  occurs outside as well as before the integral sign and the variable  $x$  appears as one of the limits of the integral. Dirichlet's problem, which is the determination of a function  $\psi$  having prescribed values over a certain boundary surface  $S$  and satisfying Laplace equation  $\nabla^2 \psi = 0$  within the region enclosed by  $S$ , was shown by Heumann in 1870 to be equivalent to the solution of an integral equation. He solved the integral equation by an expansion in powers of a certain parameter  $\lambda$ . In 1896, Italian mathematician V. Volterra gave the first general treatment of the solution of the class of linear integral equation bearing his name and characterized by the variable  $x$  appearing as the upper limit of the integral. In 1900, Swedish mathematician I. Fredholm discussed a more general class of linear integral equation having the form

$$\varphi(x) = f(x) + \lambda \int_a^b k(x,t) \varphi(t) dt,$$

The domain of the integral equation is growing up very quickly. A lot of mathematical papers and practical trials are published every month (Mikhlin (1957); Pogorzelski (1966); Widom (1969); Muskhelishvili (1953)).

In this paper, we have included a relatively new technique namely, the Hilbert formula for determining the solution of the Hilbert singular integral equation of first and second kind.

### Materials and Method

In this study, we have used the Hilbert formula for determining the solution of the Hilbert singular integral equation of first and second kind.

**Integral equation:** An integral equation is an equation in which an unknown function is to be determined appears under one or more integral signs.

Examples: 1.  $\varphi(x) = F(x) + \lambda \int_a^b k(x,t) \varphi(t) dt$  and

$$2. \varphi(x) = \lambda \int_a^b k(x,t) \varphi(t) dt.$$

The function  $\varphi(x)$  in examples (1) and (2) as the unknown function to be determined, while all other functions are known functions. The function  $k(x,t)$  is called the kernel of the integral equation and  $\lambda$  is non-zero real or complex parameter.

If  $\Omega$  is the domain of the variable  $t$ , we are often write an integral equation as

$$\varphi(x) = F(x) + \lambda \int_{\Omega} k(x,t) \varphi(t) dt.$$

**Fredholm integral equation:** An integral equation is said to be a Fredholm integral equation, if the domain of integration is fixed.

$$\text{Example: } \alpha(x) \varphi(x) = F(x) + \lambda \int_a^b k(x,t) \varphi(t) dt \quad (1)$$

(i) If  $\alpha(x) = 0$ , then equation (1) reduces to  $F(x) = \lambda \int_a^b k(x,t) \varphi(t) dt$ , this equation is said to be a Fredholm integral equation of first kind. The functions  $k(x,t)$  and  $F(x)$  and the limit  $a$  and  $b$  are

known. It is proposed to determine the unknown functions  $\varphi(x)$  so that (1) is satisfied for all values of  $x$  in the closed interval  $a \leq x \leq b$  and  $k(x, t)$  is the kernel of this equation.

(ii) If  $\alpha(x) = 1$ , then equation (1) reduces to  $\varphi(x) = F(x) + \lambda \int_a^b k(x, t)\varphi(t)dt$ , this equation is said to be a Fredholm integral equation of second kind.

(iii) If  $F(x) = 0$  and  $\alpha(x) = 1$ , then equation (1) reduces to  $\varphi(x) = \lambda \int_a^b k(x, t)\varphi(t)dt$ , this equation is said to be a homogeneous Fredholm integral equation of second kind.

**Singular integral equation:** An integral equation of the form

$$\varphi(x) = f(x) + \lambda \int_C k(x, t)\varphi(t)dt \quad (\text{Chakrabarti (1986); Peters (1972)})$$

is said to be singular if the range of integration is infinite. i.e., defined in the range  $0 < x < \infty$  or  $-\infty < x < \infty$  or in which the kernel is discontinuous, i.e., if the kernel is not square integrable.

Examples: 1.  $f(x) = \int_0^{\infty} \sin(x-t)\varphi(t)dt$

2.  $f(x) = \int_0^{\infty} e^{-xt}\varphi(t)dt$  and

3.  $f(x) = \int_0^x \frac{\varphi(t)}{\sqrt{x-t}}dt$ .

In examples (1) and (2), the range of integration is infinite, while in example (3) the range of integration is finite but the kernel is unbounded.

**The Hilbert Formula:** The Hilbert Formula is given by

$$\frac{1}{4\pi^2} \int_0^{2\pi} \cot\left(\frac{\sigma-x}{2}\right) \left\{ \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-\sigma}{2}\right) dt \right\} d\sigma = -\varphi(x) + \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) dt \quad [\text{Pennline (1976)}] \quad (2)$$

**Solution of the Hilbert singular integral equation of first kind:** Consider the equation of the first kind such as

$$f(x) = \frac{b}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt, \quad (3)$$

where  $b$  is any constant.

Firstly, we shall show that  $f(x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt$  (4)

Changing the variable  $x$  to  $t$  and  $t$  to  $\sigma$  in (4), we get

$$f(t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\sigma) \cot\left(\frac{\sigma-t}{2}\right) d\sigma \quad (5)$$

Multiplying both sides of (5) by  $\frac{1}{2\pi} \cot\left(\frac{t-x}{2}\right)$  and then integrating with respect to  $t$  from 0 to  $2\pi$ , we get

$$\frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt = \frac{1}{4\pi^2} \int_0^{2\pi} \cot\left(\frac{t-x}{2}\right) \left\{ \int_0^{2\pi} \varphi(\sigma) \cot\left(\frac{\sigma-t}{2}\right) d\sigma \right\} dt$$

Using the Hilbert formula, we get

$$\frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt = -\varphi(x) + \frac{1}{2\pi} \int_0^{2\pi} \varphi(\sigma) d\sigma$$

$$\text{or } -F(x) = -\varphi(x) + \frac{1}{2\pi} \int_0^{2\pi} \varphi(\sigma) d\sigma$$

$$\text{or } \varphi(x) = F(x) + \frac{1}{2\pi} \int_0^{2\pi} \varphi(\sigma) d\sigma \quad (6)$$

which is the Fredholm integral equation of second kind. Also where

$$F(x) = -\frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \quad (7)$$

For solution of equation (6),

$$\text{let } C = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\sigma) d\sigma$$

$$\text{or } C = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) dt \quad (8)$$

Then equation (6) can be expressed as

$$\varphi(x) = F(x) + C \quad (9)$$

$$\text{or } \varphi(t) = F(t) + C \quad (10)$$

Using equation (10) in equation (8), we get

$$\int_0^{2\pi} F(t) dt = 0$$

$$\text{or } \int_0^{2\pi} F(x) dx = 0 \quad (11)$$

By virtue of relation (6), it follows that (11) holds for all values of the function  $f(x)$ . Hence  $C$  must be an arbitrary constant and thus we find that infinite number of solutions of equation (4) exist and are given by equation (9),

$$\text{i.e. } \varphi(x) = F(x) + C$$

$$= C - \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \quad [\text{By using (7)}] \quad (12)$$

Substituting (12) in (4), we observe that  $f(x)$  given by (12) satisfies (4) if and only if

$$\int_0^{2\pi} f(x) dx = 0 \quad (13)$$

which is the necessary and sufficient condition for the Hilbert singular integral equation of the first kind, i.e. (4) to possess a solution is that condition (13) must hold good. Now, we shall proceed to find the solution of the integral equation (3).

Equation (3) can be written as

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_0^{2\pi} b\varphi(t) \cot\left(\frac{t-x}{2}\right) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \psi(t) \cot\left(\frac{t-x}{2}\right) dt \end{aligned} \quad (14)$$

where  $\psi(t) = b\varphi(t)$  (15)

Now proceeding as above, we get

$$\psi(x) = C - \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \quad (16)$$

Now, from equation (15), we get

$$\psi(x) = b\varphi(x) \quad (17)$$

Substituting  $\psi(x)$  from equation (16) in equation (17), we get

$$\varphi(x) = C_1 - \frac{1}{2\pi b} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \quad (18)$$

where  $C_1 = \frac{C}{b}$  is another arbitrary constant.

Therefore (18) gives the required solution of the equation (3).

**Solution of the Hilbert singular integral equation of second kind:** Consider the equation of the second kind such as

$$a\varphi(x) = f(x) - \frac{b}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt \quad (19)$$

where  $a$  and  $b$  are complex constants.

Equation (19) can be written as

$$f(x) = a\varphi(x) + \frac{b}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt \quad (20)$$

Define an operator  $L$  such that

$$L\varphi = a\varphi(x) + \frac{b}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt \quad (21)$$

From (20) and (21), we get

$$L\varphi = f(x) \quad (22)$$

Considering an adjoint operator  $M$  such that

$$Mg = ag(x) - \frac{b}{2\pi} \int_0^{2\pi} g(t) \cot\left(\frac{t-x}{2}\right) dt \quad (23)$$

From (22), we have

$$MLy = Mf$$

$$\text{or } M \left[ a\varphi(x) + \frac{b}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt \right] = Mf \quad (24)$$

$$\text{Let } g(x) = a\varphi(x) + \frac{b}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt \quad (25)$$

Therefore

$$g(x) = a\varphi(x) + \frac{b}{2\pi} \int_0^{2\pi} \varphi(\sigma) \cot\left(\frac{\sigma-x}{2}\right) d\sigma$$

$$\text{or } g(t) = a\varphi(t) + \frac{b}{2\pi} \int_0^{2\pi} \varphi(\sigma) \cot\left(\frac{\sigma-t}{2}\right) d\sigma \quad (26)$$

From (24) and (25), we have

$$Mg = Mf$$

$$\text{or } ag(x) - \frac{b}{2\pi} \int_0^{2\pi} g(t) \cot\left(\frac{t-x}{2}\right) dt = af(x) - \frac{b}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \quad (27)$$

$$\text{Let } F(x) = af(x) - \frac{b}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \quad (28)$$

Then (27) gives

$$ag(x) - \frac{b}{2\pi} \int_0^{2\pi} g(t) \cot\left(\frac{t-x}{2}\right) dt = F(x) \quad (29)$$

Now, using (25) and (26) in (29), we get

$$a^2\varphi(x) + \frac{ab}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt - \frac{ab}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt - \frac{b^2}{4\pi^2} \int_0^{2\pi} \cot\left(\frac{t-x}{2}\right) \left[ \int_0^{2\pi} \varphi(\sigma) \cot\left(\frac{\sigma-t}{2}\right) d\sigma \right] dt = F(x)$$

Now using the Hilbert formula, we have

$$a^2\varphi(x) - b^2 \left[ -\varphi(x) + \frac{1}{2\pi} \int_0^{2\pi} u(\sigma) d\sigma \right] = F(x)$$

$$\text{or } \varphi(x) = \frac{1}{a^2 + b^2} F(x) + \frac{b^2}{2\pi(a^2 + b^2)} \int_0^{2\pi} \varphi(t) dt \quad (30)$$

which is the Fredholm integral equation of second kind with separate Kernel.

Now, we want to find the solution of (30)

$$\text{Let } C = \int_0^{2\pi} \varphi(t) dt \quad (31)$$

Then (30) gives

$$\varphi(x) = \frac{1}{a^2 + b^2} F(x) + \frac{b^2}{2\pi(a^2 + b^2)} \cdot C \quad (32)$$

$$\text{or } \varphi(t) = \frac{1}{a^2 + b^2} F(t) + \frac{b^2 C}{2\pi(a^2 + b^2)} \quad (33)$$

Putting the value of  $\varphi(t)$  in (31), we get

$$C = \frac{1}{a^2} \int_0^{2\pi} F(t) dt$$

Thus we have  $C = \frac{1}{a} \int_0^{2\pi} f(t) dt$        $[\ominus F(t) = af(t)]$

Putting the value of  $C$  in (32), we get

$$\begin{aligned} \varphi(x) &= \frac{F(x)}{a^2 + b^2} + \frac{b^2}{2\pi(a^2 + b^2)} \cdot \frac{1}{a} \int_0^{2\pi} f(t) dt \\ &= \frac{1}{a^2 + b^2} \left[ af(x) - \frac{b}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \right] + \frac{b^2}{2\pi a(a^2 + b^2)} \int_0^{2\pi} f(t) dt \\ \therefore \varphi(x) &= \frac{a}{a^2 + b^2} \cdot f(x) - \frac{b}{2\pi(a^2 + b^2)} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt + \frac{b^2}{2\pi a(a^2 + b^2)} \int_0^{2\pi} f(t) dt \end{aligned} \quad (34)$$

which is the required solution.

### Results

The solution of the Hilbert singular integral equation is also a Hilbert-type after solving by the Hilbert formula because the Hilbert kernel  $\cot\left(\frac{t-x}{2}\right)$  exists in the solution of the Hilbert-type singular integral equation and the Hilbert formula generates both the solution and the original Hilbert-type singular integral equation of first kind from the Hilbert-type singular integral equation of second kind after applying a particular condition on the Hilbert-type singular integral equation of second kind.

### Discussion

In this section, we have considered some interesting generalizations applying different conditions to the equation (3) and (19).

**Generalization 1:** The Hilbert-type singular integral equation of first kind is given by (3) as

$$f(x) = \frac{b}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt \quad (E1)$$

and its solution is given by (18) as

$$\varphi(x) = C_1 - \frac{1}{2\pi b} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \quad (E2)$$

If we consider  $b = 1$  in (E1), then equation (E1) becomes

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt \quad (E3)$$

and its solution is given by

$$\varphi(x) = C_1 - \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt \quad (E4)$$

which has infinite number of solution because  $C_1$  is an arbitrary constant.

**Generalization 2:** The Hilbert-type singular integral equation of second kind is given by (19) as

$$a\varphi(x) = f(x) - \frac{b}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt \quad (E5)$$

and its solution is given by (34) as

$$\therefore \varphi(x) = \frac{a}{a^2 + b^2} \cdot f(x) - \frac{b}{2\pi(a^2 + b^2)} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt + \frac{b^2}{2\pi a(a^2 + b^2)} \int_0^{2\pi} f(t) dt \quad (E6)$$

**Case 1:**

If we consider  $a = 0$  in (E5), then equation (E5) reduces to the first kind as

$$f(x) = \frac{b}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt \quad (E7)$$

and its solution is given by

$$\varphi(x) = \frac{1}{b^2} \cdot f(x) - \frac{1}{2\pi b} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt + \frac{1}{2\pi i} \int_0^{2\pi} f(t) dt \quad (E8)$$

**Case 2:**

If  $a = 0$  and  $b = 1$  in (E5), then equation (E5) reduces to

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \cot\left(\frac{t-x}{2}\right) dt \quad (E9)$$

and its solution is given by

$$\varphi(x) = f(x) - \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{t-x}{2}\right) dt + \frac{1}{2\pi i} \int_0^{2\pi} f(t) dt \quad (E10)$$

**Conclusion**

From our above discussion we have seen that the Hilbert formula (HF) is applicable for both the Hilbert-type singular integral equation of first and second kind whose kernel is of the form  $k(x,t) = \cot\left(\frac{t-x}{2}\right)$ . Also, we see that the Hilbert formula generates both the solution and the original Hilbert-type singular integral equation of first kind from the Hilbert-type singular integral equation of second kind after applying a particular condition on the Hilbert-type singular integral equation of second kind. So, from our point of view, we can say that the another name of the Hilbert formula is generator or converter of the Hilbert-type singular integral equation of first kind.

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