



ASYMPTOTIC SOLUTION OF SECOND ORDER OVER-DAMPED NON-LINEAR SYSTEMS FOR SMALL RATIO OF THE EIGENVALUES

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Abstract: A technique is developed for obtaining the solution of second order over-damped non-linear systems based on the Krylov-Bogoliubov-Mitropolskii (KBM) method for small ratio of the eigenvalues. The solution is also useful for large ratio of the eigenvalues. The method is illustrated by an example.

Key words: Perturbation, Asymptotic Solution, Over-Damped

Introduction

Most of the perturbation methods were developed to find periodic solution of non-linear systems, but Krylov-Bogoliubov (1947) introduced a perturbation method to discuss the transients of second order nonlinear systems $[\ddot{x} + \omega_0^2 x + \varepsilon f(x, \dot{x}) = 0]$ with small nonlinearities. Then Bogoliubov-Mitropolskii (1961) amplified and justified this method. Thus the method is named the Krylov-Bogoliubov-Mitropolskii (KBM) method. Later, Popov (1956) extended this method to solve damped oscillatory non-linear systems $[\ddot{x} + 2k\dot{x} + \omega^2 x + \varepsilon f(x, \dot{x}) = 0, \quad k > 0]$ in presence of strong linear damping forces. Mendelson (1970) and Bojadziev (1983) rediscovered Popov's results owing to physical importance. For finding the solution of over-damped nonlinear systems, followed by Popov's technique, Murty *et al.* (1969) first extended the KBM method. Murty (1971) also presented a unified KBM method to solve a second order nonlinear system which covers the un-damped, damped and over-damped cases. Sattar (1986) extended the KBM asymptotic method for solving second order critically damped non-linear systems. Later, Alam (2001) found an asymptotic solution for second order over-damped and critically damped nonlinear systems. Alam (2003) again investigated the solution of second order over-damped non-linear systems under some special conditions. But Alam's (2003) solution gives desired results only for large ratio of the eigenvalues. In the case of small ratio of the eigenvalues, Alam's (2003) solution fails to give desired results.

In this article, we have developed a technique which gives desired results for both the cases, when the ratio of the eigenvalues is small as well as when the ratio of the eigenvalues is large.

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Materials and Methods

Consider a quasi-linear over-damped system governed by the second order differential equation

$$\ddot{x} + k_1 \dot{x} + k_2 x + \varepsilon f(x, \dot{x}) = 0 \quad (1)$$

where $\dot{x}$ and $\ddot{x}$ denote the derivatives of x respect to t , ε is a small parameter, k_1 and k_2 are constants and f is the non-linear function. As the system is over-damped, the eigenvalues of the corresponding linear equation are real, unequal and negative. Let us consider, the eigenvalues are $-\lambda$ and $-\mu$. When $\varepsilon = 0$, the solution of the corresponding linear equation of (1) is

$$x(t, 0) = a e^{-\lambda t} + b e^{-\mu t}, \quad (2)$$

where a and b are integrating constants.

When $\varepsilon \neq 0$, following KBM (1947, 1961) method, we seek a solution of (1) in the form

$$x(t, \varepsilon) = a(t) e^{-\lambda t} + b(t) e^{-\mu t} + \varepsilon u_1(a, b, t) + \varepsilon^2 u_2(a, b, t) + K, \quad (3)$$

where a and b satisfy the first order differential equations

$$\left. \begin{aligned} \dot{a} &= \varepsilon A_1(a, b, t) + \Lambda \\ \text{and } \dot{b} &= \varepsilon B_1(a, b, t) + \Lambda \end{aligned} \right\}. \quad (4)$$

Confining only to the first few terms $1, 2, \dots, m$ in the series expansions of (3) and (4), we evaluate the functions A_1 , B_1 and u_i , $i = 1, 2, 3, K, m$ such that A_1 and B_1 appearing in (3) and (4) satisfy the given differential equation (1) with an accuracy of order ε^{m+1} . In order to determine these unknown functions it is customary in the KBM method that the correction terms u_i must exclude those terms (known as secular terms) which make them large. Theoretically, the solution can obtain up to the accuracy of any order of approximation. But owing to rapidly growing algebraic complexity for the derivation of the formulae, we have confined to the lower order (see also Murty, 1971 for details).

Differentiating (3) twice with respect to t , substituting (3) and the derivatives in the original equation (1), then using relations of (4), and finally equating the coefficients of ε , we obtain

$$e^{-\lambda t} (D + \mu - \lambda) A_1 + e^{-\mu t} (D + \lambda - \mu) B_1 + (D + \lambda)(D + \mu) u_1 = -f^{(0)}(a, b, t), \quad (5)$$

where $f^{(0)} = f(x_0, \dot{x}_0)$ and $x_0 = a e^{-\lambda t} + b e^{-\mu t}$.

In general, $f^{(0)}$ can be expanded in Taylor series (see also Murty, 1969; Alam, 2001, 2003) as:

$$f^{(0)} = \sum_{i,j=0}^{\infty} F_{i,j} e^{-(i\lambda+j\mu)t}. \quad (6)$$

Substituting the value of $f^{(0)}$ in (5), we obtain

$$e^{-\lambda t} (D + \mu - \lambda) A_1 + e^{-\mu t} (D + \lambda - \mu) B_1 + (D + \lambda)(D + \mu) u_1 = - \sum_{i,j=0}^{\infty} F_{i,j} e^{-(i\lambda+j\mu)t}. \quad (7)$$

In this article, we have introduced a new condition that u_1 contains the terms $e^{-(i\lambda+j\mu)t}$, where $i = j$ or $i = 0$ or $j = 0$, to solve the equation (7) for unknown functions A_1 , B_1 and u_1 . If the ratio of the two eigenvalues is not large, under this condition the coefficients of terms u_1 do not become large. Therefore, under this new assumption, we obtain

$$e^{-\lambda t} (D + \mu - \lambda) A_1 + e^{-\mu t} (D + \lambda - \mu) B_1 = - \sum_{i,j=0}^{\infty} F_{i,j} e^{-(i\lambda+j\mu)t} \quad (8)$$

when $i \neq j$ or $i \neq 0$ or $j \neq 0$

$$\text{and } (D + \lambda)(D + \mu) u_1 = - \sum_{i,j=0}^{\infty} F_{i,j} e^{-(i\lambda+j\mu)t} \quad (9)$$

when $i = j$ or $i = 0$ or $j = 0$.

Now, we have only one equation to find the unknown functions A_1 and B_1 . The value of A_1 and B_1 can be found subject to the condition that the coefficients of A_1 and B_1 do not become large for any time t . Solving equation (9) the value of u_1 can be found. This completes the determination of the solution of the system (1).

Example: As an example of the above method, we have considered a quasi-linear second order differential system

$$\ddot{x} + k_1 \dot{x} + k_2 x = -\varepsilon x^3, \quad (10)$$

where ε is a small parameter, k_1 and k_2 are constants and $f = x^3$.

Therefore,

$$f^{(0)} = a^3 e^{-3\lambda_1 t} + 3a^2 b e^{-(2\lambda+\mu)t} + 3ab^2 e^{-(\lambda+2\mu)t} + b^3 e^{-3\mu t}. \quad (11)$$

For equation (10) and for our new assumption, equations (8) and (9) respectively become

$$e^{-\lambda t} (D + \mu - \lambda) A_1 + e^{-\mu t} (D + \lambda - \mu) B_1 = - \left(3a^2 b e^{-(2\lambda+\mu)t} + 3ab^2 e^{-(\lambda+2\mu)t} \right) \quad (12)$$

$$\text{and } (D + \lambda)(D + \mu)u_1 = - \left(a^3 e^{-3\lambda t} + b^3 e^{-3\mu t} \right). \quad (13)$$

Solving equation (13), we arrive at

$$u_1 = \frac{a^3 e^{-3\lambda t}}{2\lambda(\mu - 3\lambda)} + \frac{b^3 e^{-3\mu t}}{2\mu(\lambda - 3\mu)}. \quad (14)$$

Now, we have only one equation (12) for obtaining two unknown functions A_1 and B_1 . Since the ratio of the two eigenvalues are small, so, to split the equation (12) for determining the unknown functions A_1 and B_1 , we have used the restriction that, if $i \geq j$, the term $e^{-(i\lambda+j\mu)t}$ balance with A_1 , and if $j > i$ the term $e^{-(i\lambda+j\mu)t}$ balance with B_1 (see also Islam *et al.* 2007 for details).

Under this condition, we obtain from equation (12)

$$e^{-\lambda t} (D + \mu - \lambda) A_1 = -3a^2 b e^{-(2\lambda+\mu)t} \quad (15)$$

$$\text{and } e^{-\mu t} (D + \lambda - \mu) B_1 = -3a^2 b e^{-(\lambda+2\mu)t}. \quad (16)$$

Solving equations (15) and (16), we obtain

$$A_1 = \frac{3a^2 b e^{-(\lambda+\mu)t}}{2\lambda} \quad \text{and} \quad B_1 = \frac{3ab^2 e^{-(\lambda+\mu)t}}{2\mu}. \quad (17)$$

Substituting the values of A_1 and B_1 into the equation (4), we obtain

$$\ddot{x} = \frac{3\varepsilon a^2 b e^{-(\lambda+\mu)t}}{2\lambda} \quad \text{and} \quad \ddot{x} = \frac{3\varepsilon a b^2 e^{-(\lambda+\mu)t}}{2\mu}. \quad (18)$$

Since ε is small, so we can solve equation (18), by assuming a and b are constants in the right hand side of (18). This assumption was first made by Murty *et al.* (1969) to solve similar type of non-linear equations. Thus, the solutions of (18) is

$$a = a_0 + \varepsilon \frac{3a_0^2 b_0 (1 - e^{-(\lambda+\mu)t})}{2\lambda(\lambda + \mu)} \quad (19)$$

$$\text{and } b = b_0 + \varepsilon \frac{3a_0 b_0^2 (1 - e^{-(\lambda+\mu)t})}{2\mu(\lambda + \mu)}.$$

Therefore, the first approximate solution of (10) is

$$x(t, \varepsilon) = a e^{-\lambda t} + b e^{-\mu t} + \varepsilon u_1, \quad (20)$$

where a and b are given by (19) and u_1 is given by (14).

Discussion of Alam (2003): Alam (2003) considered the same second order non-linear differential system as we have considered in equation (1) and same example as we have considered in equation (10).

To solve equation (7) for A_1 , B_1 and u_1 , Alam (2003) introduced the condition that u_1 contains the terms with $e^{-(j\lambda+k\mu)t}$, where $j\lambda+k\mu > \frac{1}{2}(j+k)(\lambda+\mu)$, so that the coefficients of terms of u_1 do not become large or u_1 does not contain secular type terms $t e^{-t}$. According to Alam's condition, for example (10), Alam obtained

$$e^{-\lambda t}(D+\mu-\lambda)A_1 + e^{-\mu t}(D+\lambda-\mu)B_1 = -(3ab^2e^{-(\lambda+2\mu)t} + b^3e^{-3\mu t}) \quad (21)$$

$$\text{and } (D+\lambda)(D+\mu)u_1 = -(a^3e^{-3\lambda t} + 3a^2be^{-(2\lambda+\mu)t}). \quad (22)$$

Solving equation (22), Alam obtained

$$u_1 = \left(\frac{a^3e^{-3\lambda t}}{2\lambda(3\lambda-\mu)} + \frac{a^2be^{-(2\lambda+\mu)t}}{2\lambda(\lambda+\mu)} \right). \quad (23)$$

To determine the unknown functions A_1 and B_1 from the single equation (21), Alam considered that $\lambda \gg \mu$ or $\lambda+2\mu \approx \lambda$. i. e. the ratio of the eigenvalues is very large. Therefore, from equation (21) he obtained

$$\frac{\partial A_1}{\partial t} + \mu A_1 - \lambda A_1 = -3ab^2e^{-2\mu t} \quad (24)$$

$$\text{and } \frac{\partial B_1}{\partial t} + \lambda B_1 - \mu B_1 = -b^3e^{-2\mu t}.$$

Solving (24), Alam's obtained

$$A_1 = \frac{3ab^2e^{-2\mu t}}{(\lambda+\mu)} \quad (25)$$

$$\text{and } B_1 = \frac{b^3e^{-2\mu t}}{3\mu-\lambda}.$$

Substituting the values of A_1 and B_1 in (4) and solving, Alam obtained

$$a = a_0 + \frac{3\varepsilon a_0 b_0^2(1-e^{-2\mu t})}{2\mu(\lambda+\mu)} \quad (26)$$

$$\text{and } b = \frac{b_0}{\sqrt{1 + \frac{\varepsilon b_0^2(e^{-2\mu t}-1)}{\mu(3\mu-\lambda)}}}.$$

Thus, Alam obtained the first approximate solution of the example (10) as

$$x(t, \varepsilon) = ae^{-\lambda t} + be^{-\mu t} + \varepsilon u_1, \quad (27)$$

where a and b are given by the equation (26) and u_1 is given by the equation (23).

Results

In order to test the accuracy, we compare the approximate solution to the numerical solution.

Regarding to such a comparison concerning the presented KBM method of this article, we refer to works of Murty *et al.* (1969) and Mendelson (1970).

First of all, we have chosen $\lambda = 3.0$ and $\mu = 0.75$. i. e. the ratio of the eigenvalues is 4. We have computed $x(t, \varepsilon)$ by the equation (20) in which a and b are computed by the equation (19) together with initial conditions $a_0 = 0.5$ and $b_0 = 0.1$ for various values of t when $\varepsilon = 0.1$ and the results are presented in the second column of the Table 1. Corresponding numerical solutions (designated by x_{nu}) have also been computed by a fourth order Runge-Kutta method and the

results are presented in the third column of the Table 1. Then $x(t, \varepsilon)$ has been computed (designated by x_{SA}) by Alam's (2003) solution (27), where a and b are computed by the equation (26) and the results are presented in the fifth column of the Table 1. Corresponding numerical solutions (designated by x_{nuSA}) have also been computed by a fourth order Runge-Kutta method and the results are presented in the sixth column of the Table 1. Column four and seven show the percentage errors of ours and Alam (2003), respectively.

Table 1. Comparison of ours and Alam's (2003) results when $\lambda = 3.0$ and $\mu = 0.75$

t	x	x_{nu}	$E\%$	x_{SA}	x_{nuSA}	$E_{SA}\%$
0.0	0.599836	0.599836	0.000	0.599414	0.599414	0.000
0.5	0.180538	0.181878	0.736	0.180279	0.182497	1.121
1.0	0.072279	0.072930	0.892	0.072107	0.073561	1.976
1.5	0.038113	0.038382	0.700	0.037997	0.038861	2.223
2.0	0.023614	0.023739	0.526	0.023534	0.024079	2.263
2.5	0.015653	0.015727	0.470	0.015599	0.015963	2.280
3.0	0.010630	0.010681	0.477	0.010592	0.010844	2.323
3.5	0.007277	0.007316	0.533	0.007251	0.007428	2.382
4.0	0.004995	0.005025	0.597	0.004977	0.005102	2.450
4.5	0.003432	0.003454	0.636	0.003419	0.003507	2.509
5.0	0.002358	0.002375	0.715	0.002350	0.002412	2.570

Then, we have chosen $\lambda = 3.0$ and $\mu = 0.5$. i. e. the ratio of the eigenvalues is 6. In this case, the results are presented in the respective columns of the Table 2.

Table 2. Comparison of ours and Alam's (2003) results when $\lambda = 3.0$ and $\mu = 0.5$

t	x	x_{nu}	$E\%$	x_{SA}	x_{nuSA}	$E_{SA}\%$
0.0	0.599822	0.599822	0.000	0.599398	0.599398	0.000
0.5	0.189816	0.191131	0.688	0.189446	0.191675	1.162
1.0	0.085831	0.086454	0.720	0.085534	0.087074	1.768
1.5	0.053004	0.053249	0.460	0.052770	0.053776	1.870
2.0	0.038189	0.038299	0.287	0.038007	0.038719	1.838
2.5	0.029052	0.029117	0.223	0.028910	0.029447	1.823
3.0	0.022471	0.022521	0.222	0.022361	0.022778	1.830
3.5	0.017466	0.017508	0.239	0.017380	0.017709	1.857
4.0	0.013595	0.013631	0.264	0.013528	0.013787	1.878
4.5	0.010586	0.010617	0.291	0.010534	0.010739	1.908
5.0	0.008244	0.008271	0.326	0.008203	0.008366	1.948

Finally, we have chosen $\lambda = 3.0$ and $\mu = 0.25$. i. e. the ratio of the eigenvalues is 12. In this case, also the results are presented in the respective columns of the Table 3.

Table 3. Comparison of ours and Alam's (2003) results when $\lambda = 3.0$ and $\mu = 0.25$

t	x	x_{nu}	$E\%$	x_{SA}	x_{nuSA}	$E_{SA}\%$
0.0	0.599851	0.599851	0.000	0.599377	0.599377	0.000
0.5	0.200598	0.201892	0.640	0.199823	0.202071	1.112
1.0	0.103526	0.104120	0.570	0.102764	0.104413	1.579
1.5	0.074947	0.075162	0.286	0.074257	0.075451	1.582
2.0	0.062473	0.062555	0.131	0.061860	0.062816	1.521
2.5	0.054311	0.054352	0.075	0.053769	0.054585	1.494
3.0	0.047744	0.047774	0.062	0.047266	0.047980	1.488
3.5	0.042091	0.042119	0.066	0.041669	0.042300	1.491
4.0	0.037135	0.037161	0.069	0.036763	0.037322	1.497
4.5	0.032769	0.032794	0.076	0.032440	0.032936	1.509
5.0	0.028917	0.028942	0.086	0.028627	0.029067	1.513

Discussion

The results of the Table 1 are obtained for $\lambda = 3.0$ and $\mu = 0.75$. i. e. when the ratio of the eigenvalues is 4. From Table 1, we see that our percentage errors are smaller than 1%. On the other hand, the percentage errors of Alam's (2003) exceed 2%. Again, the results of the Table 2 are obtained for $\lambda = 3.0$ and $\mu = 0.5$. i. e. when the ratio of the eigenvalues is 6. From Table 2, we see that our percentage errors are much smaller than 1%. But the percentage errors of Alam's (2003) are near about 2%. Therefore, from Table 1 and Table 2, we observe that when the ratio of the eigenvalues increases, the percentage errors decrease. To clarify this claim, we have again chosen $\lambda = 3.0$ and $\mu = 0.25$. i. e. the ratio of the eigenvalues is 12 and the results are presented in the Table 3. From Table 3, we see that our percentage errors are smaller than 1%. On the contrary, the percentage errors of Alam (2003) exceed 1% in this case also. From all the Tables, we see that our results always show the good congruency with the numerical results, but Alam's (2003) solution gives results with errors more than 2% and the errors decrease with the increase of the ratio of the eigenvalues. Thus, Alam's (2003) solution is useful only when the ratio of the eigenvalues is large, but our solution give desire results both for small and large ratio of the eigenvalues.

Conclusion

A perturbation technique is developed in this article to obtain the solution of second order over-damped non-linear systems for small ratio of the eigenvalues. The solution is also useful for large ratio of the eigenvalues. This is an extension of the Krylov-Bogoliubov-Mitropolskii (KBM) method.

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