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A REVIEW OF SYMMETRIES, CONSERVED QUANTITIES AND INVARIANCE PROPERTIES

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Abstract: Symmetries, conserved quantities and invariance properties are studied. In the case of Lagrangian systems the connection between conserved quantities and dynamical symmetries is not too direct, for instance, there always exists an infinite number of linearly independent dynamical symmetries which have no associated Neother constant of the motion but for general systems dynamical symmetries always possesses associated conserved quantities which are invariants of the symmetry group itself. Now a days symmetry properties are crucial for renormalizability of a theory under consideration which helps in characterizing the above mentioned features of the micro world and is a tool for exploration of further possibility. In this study we have made an attempt to discuss the relationship among symmetry, conserved quantity and invariance property with their applications.

Key words: Physical laws, Lagrangian systems, Neother constant, manifolds, Lie Groups

Introduction

A recurring theme throughout has been that the symmetry properties of the Lagrangian or Hamiltonian imply the existence of conserved quantities. Thus, if the Lagrangian does not contain explicitly a particular coordinate of displacement, then the corresponding canonical momentum is conserved. The absence of explicit dependence on the coordinate means the Lagrangian is unaffected by a transformation that alters the value of that coordinate; it is said to be invariant, or symmetric under the given transformation. Similarly, invariance of the Lagrangian under time displacement implies conservation of energy. The formal description of the connection between invariance or symmetry properties and conserved quantities is contained in Neother's theorem (Goldstein, 1950).

The earliest examples of a documented application of symmetry principles are no doubt the use of patterns of bilateral and/or periodic symmetry in decorative and monumental art, going back at least to the Sumerians (2700 BC). Much later several schools of Islamic art used abstract symmetric patterns to great effect, reportedly employing all the 17 symmetry types of space groups in 2 dimensions. The decorative aspect of symmetry is not the most important in the present context but it allows us to introduce some basic mathematics without worrying about the physics (Jablan, 1995).

There are special systems in non-homogenous and non-isotropic environments which have a more restricted symmetry. For instance, there is the spherical symmetry (rotation invariance) of the Kepler problem of a light planet moving in the central force field of a heavy sun. The conservation of angular momentum is Kepler's second law. Particles moving in a periodic potential is another example, the symmetry group now contains a discrete group of translations.

Neother (1918a) proved a general theorem connecting symmetries with conserved quantities, or integral of the motion, as they often called in Classical Mechanics. That is, there is a close connection between the

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conserved quantities of a Lagrangian dynamical system and those transformations of coordinates and time which preserve the action integral. Transformations which leave the action invariant form a Lie group (Lie, 1967), which itself may be a proper subgroup of the group of transformations leaving the equation of motion invariant (Lutzky, 1978).

The invariance of dynamical systems under more general transformations, which allow the new coordinates and time to depend on the old velocities, as well as on the old coordinates and time. Such transformations operate on trajectories in space-time, and will be called dynamical symmetries if they preserve the equations of motion. The invariance properties of differential equations under derivative-dependent transformations have been studied by several authors (Ovsjannikov, 1969; Anderson *et al.,* 1976); in particular Anderson *et al*. (1976) provided a treatment which may be applied to second order dynamical systems. Their methods constitute an extension of the Ovsjannikov theory, which is itself a generalization of the Lie Theory of extended groups (Cohen, 1931). We call any transformation which posses an associated Neother conserved quantity a Neother transformation. It follows that a symmetry transformation need not be a Neother transformation, and a Neother transformation need not be a symmetry transformation. Rosen (1981) noted this in the context of classical field theory; which are quite different from us.

Preliminaries

Most studies conducted on symmetries, conserved quantities and invariance properties based on Neother's celebrated theorem (Neother, 1918a). We take a review of Classical mechanics in which Lagrangian and Hamiltonian equation of motion is included (Katz, 1965). We also take a review of Group theory and its applications (Inui *et al*., 1996), symmetry properties (Ludwig and Falter, 1996). In this study we derive a condition for invariance transformation in a different manner than that employed Anderson *et al.* (1976); the criterion used here will be that conserved quantities must be transformed into conserved quantities. Finally we demonstrate that our results apply also to the special case of point symmetries; such as the conserved quantities associated with a point symmetry group are invariants of the first extended group.

Symmetry in general

In general, the symmetry properties of any things may be appeared under a system. The system may be abstract or concrete, microscopic or macroscopic, static or dynamic, etc.

A transformation of a system is a mapping of a state space of the system into itself. We can denote a transformation *T* by

$u \xrightarrow{T} v$ or, $v = T(u)$, Where u and v represent states of the system.

The transformations that map every state to an image state equivalent to the object state for a state space with equivalence relation, such transformations are called a symmetry transformation. Thus, a symmetry transformation *S* is defined as

$u \xrightarrow{s} v \equiv u$ or, $S(u) \equiv v = u$ for all states u.

The set of all invertible symmetry transformations of a state space of system for an equivalence relation forms a group, a subgroup of the transformation group, called the symmetry group of the systems for the equivalence relation.

Manifold

A set of point M is defined to be a manifold if each point of M has an open neighborhood which has a continuous one-one map onto an open set $\Re^n$ for some positive integer n.

Example: (i) The set of all rotations of a rigid object in three dimensions is a manifold, since it can be continuously parameterized by the three 'Euler angles' (Goldstein, 1950).

Lie groups

A Lie group is a differentiable manifold which is endowed with a group structure such that the group operations

(i) $o: G\times G \rightarrow G \quad defined \quad by \quad (g_1, g_2)\rightarrow g_1\, o\, g_2$

(ii) $^{-1}: G\times G \rightarrow G \quad defined \quad by \quad g \rightarrow g^{-1}$ are differentiable.

The space and time displacement groups are non-compact Lie group, and the rotation group is compact Lie group. The Lorentz group is a six parameter (three rotations, three velocities) Lie group. Example: The simplest example of a Lie group is the real line $\Re$ with ordinary addition as the group operation.

Hamilton's Principle

Hamilton's principle for conservative system is stated as follows:

The motion of the system from time t_1 to time t_2 is such that the line integral $I = \int_{t_1}^{t_2} L\, dt$ where $L = T - V$, is an extremum for the path of motion.

Neother's Theorem

If for an infinitesimal transformation of the generalized coordinates q_i of a holonomic system and of time t of the form

$$q_i' = q_i + \varepsilon \Psi_i(q_i, t)$$
$$t' = t + \varepsilon\chi(q_i, t), \quad \varepsilon \rightarrow 0 \tag{1}$$

Hamilton's principle function is invariant, that is,

$$\delta\int_{t_1}^{t_2} L(q_i, \frac{dq_i}{dt}, t)\, dt = \delta\int_{t_1'}^{t_2'} L(q_i', \frac{dq_i'}{dt'}, t')\, dt'$$ then the quantity

$$\frac{\partial L}{\partial q_i}(\dot{q}_i\chi - \Psi_i) - L\chi \tag{2}$$

is an integral of motion.

Proof: We know that the variation of Hamilton's principle function stands for the variation of the terminal coordinates and time, that is,

$$\delta W = \delta\int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = \left[\frac{\partial W}{\partial q_i}\delta q_i + \frac{\partial W}{\partial t}\delta t\right]_a^f$$

If we now interpret δq_i and δt which are infinitesimal variations or displacements in the values of the coordinates and time of the original frame, to be effectively equivalent to the ones generated by the infinitesimal coordinates and time transformations given by Eq. (1), then $\delta q_i = -\varepsilon\Psi_i(q, t)$ and $\delta t = -\varepsilon\chi(q, t)$ for all points of the path including the end points. This is called a changeover from an active to a passive viewpoint. Let us digress here for a moment in order to clarify this point with an example.

Suppose a rigid body rotates about an axis represented by the unit vector $\hat{n}$ by an amount $\Delta\theta$ in time Δt. The position vector of any particle at $\vec{r}$ will now actively change to $\vec{r} + \Delta\vec{r} = \vec{r} + (\hat{n}\times\vec{r})\Delta\theta$. But the same change in coordinates can be effected by rotating the coordinate axes about $\hat{n}$ by an angle $-\Delta\theta$. The former viewpoint is called an *active viewpoint* and the latter a *passive* one.

So, the requirements of the invariance of W under passive coordinate and time transformation given by Eq. (1) can be effectively viewed as one under an active displacement in both coordinates and time, with a change of sign. Thus, $\delta W = 0$ under the transformation Eq. (1) will effectively mean

$$\left[\frac{\partial W}{\partial q_i}\varepsilon\Psi_i(q,t)+\frac{\partial W}{\partial t}\varepsilon\chi(q,t)\right]_{final}=\left[\frac{\partial W}{\partial q_i}\varepsilon\Psi_i(q,t)+\frac{\partial W}{\partial t}\varepsilon\chi(q,t)\right]_{initil}$$

Now since ε is arbitrary, $p_i=\frac{\partial W}{\partial q_i}$ and $H=-\frac{\partial W}{\partial t}$, we must have, for all points of the path,

$$p_i\Psi_i(q,t)-H\chi(q,t)=const.$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_i}\Psi_i(q,t)-\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i-L\right)\chi(q,t)=const.$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_i}\left(\dot{q}_i\chi(q,t)-\Psi(q,t)\right)-L\chi(q,t)=const.$$

The theorem is valid for all bilateral holonomic systems. Neother's theorem in the above form can further be generalized to the case when the transformation of coordinates and time changes Hamilton's principal function in the following way;

$$\int_{t_1}^{t_2} L\left(q,\frac{dq}{dt},t\right)dt=\int_{t_1'}^{t_2'}\left[L\left(q',\frac{dq'}{dt'},t'\right)+\varepsilon\frac{dF(q',t')}{dt'}\right]dt'$$

that is when W is invariant under the Lagrangian gauge transformation, to a new set of coordinates and time. Obviously, this would simply add $F(q,t)$ to Eq. (2) giving

$$\frac{\partial L}{\partial \dot{q}_i}\left(\dot{q}_i\chi-\Psi_i\right)-L\chi+F=const. \qquad (3)$$

This is the most general form of Neother's theorem first given by Neother (1918a).

Now for any closed system, all the conservation laws due to homogeneity of space and time and isotropy of space would follow immediately from Eq. (2) and (3), as shown below.

(i) For the homogeneity of time, we put $q_i' = q_i$ and $t' = t+\varepsilon$ so that $\Psi_i = 0$ and $\chi = 1$ and Eq. (2) gives

$$\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i-L=const. \qquad (4)$$

This is the law of conservation of energy for a closed system.

(ii) For the homogeneity of space , we use Cartesian coordinates with the infinitesimal coordinates transformation $x_i' = x_i+\varepsilon$ and $t' = t$ that is $\Psi_i = 1$ and $\chi = 0$ for any particular i, giving

$$\frac{\partial L}{\partial \dot{x}_i}=const. \quad \Rightarrow p_i=const. \qquad (5)$$

So the component of p_i of the linear momentum corresponding to any Cartesian coordinate x_i is conserved for a closed system.

(iii) For the isotropy of space , we choose any particular generalized coordinate q_i as the angle θ and $q_i' = q_i + \varepsilon$, $t' = t$ with $\Psi_i = 1$ and $\chi = 0$ giving

$$\frac{\partial L}{\partial \dot{q}_i} = const. = \frac{\partial L}{\partial \dot{\theta}} \tag{6}$$

Thus the angular momentum corresponding to the θ - rotation is constant.

(iv) For invariance under the Galilean transformation, we must use the form given by Eq. (3). Taking $t' = t$, $x' = x - \varepsilon t$ so that $\chi = 0$ and $\Psi_i = -t$ and the required $F = -m\dot{x}$, for a single particle moving in the $x - direction$, Neother's theorem in the form Eq. (3) says that

$$- m\dot{x} + mx = const.$$

Actually, for a system of particles,

$$\sum m_i x_i - \sum p_i t = const. \tag{7}$$

which is known as Galilean translational invariance.

Properties of dynamical symmetries

Consider a transformation from $p+1$ variables $q_1, q_2, \ldots, q_p, t$ to the $p+1$ variables $Q_1, Q_2, \ldots, Q_p, T$, and let us suppose that the new variables also depend on the p time derivatives $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_p$. This transformation may be written

$$Q_l = Q_l(q, \dot{q}, t) \qquad T = T(q, \dot{q}, t) \tag{8}$$

Where q_l are to be considered functions of the time t. The equations $q_l = q_l(t)$ define a curve in $q,\ t$ space, with $\dot{q}_l = \frac{dq_l}{dt}$; the above transformation may therefore be said to act on curves in $q,\ t$ space, and to produce curves in $Q,\ T$ space. Expressing q_l, $\dot{q}_l$ in terms of t and eliminating t from Eq. (8) yields the resulting curve $Q_l = Q_l(T)$ in $Q,\ T$ space. The time derivatives $\dot{Q}_l = \frac{dQ_l}{dT}$ may be found in terms of $q_l, \dot{q}_l, t$ as follows;

$$\dot{Q}_l = \frac{\left(\frac{dQ_l}{dt}\right)}{\left(\frac{dT}{dt}\right)} = \frac{\left\{\frac{\partial Q_l}{\partial t} + \left(\frac{\partial Q_l}{\partial q_m}\right)\dot{q}_m + \left(\frac{\partial Q_l}{\partial \dot{q}_m}\right)\ddot{q}_m\right\}}{\left\{\frac{\partial T}{\partial t} + \left(\frac{\partial T}{\partial q_m}\right)\dot{q}_m + \left(\frac{\partial T}{\partial \dot{q}_m}\right)\ddot{q}_m\right\}} \tag{9}$$

If we assume that the curve $q_l = q_l(t)$ satisfies the following differential equation

$$\dot{q}_l = \alpha_l(q, \dot{q}, t) \qquad (l = 1, 2, \ldots \ldots, p) \tag{10}$$

then Eq. (9) may be written as follows;

$$\dot{Q}_l = \frac{\left\{ \dfrac{\partial Q_l}{\partial t} + \left(\dfrac{\partial Q_l}{\partial q_m} \right) \dot{q}_m + \left(\dfrac{\partial Q_l}{\partial \dot{q}_m} \right) \alpha_m \right\}}{\left\{ \dfrac{\partial T}{\partial t} + \left(\dfrac{\partial T}{\partial q_m} \right) \dot{q}_m + \left(\dfrac{\partial T}{\partial \dot{q}_m} \right) \alpha_m \right\}} \tag{11}$$

Equations (8) and (11) continue a transformation from $(q, \dot{q}, t)$ variables to the $(Q, \dot{Q}, T)$ variables; this transformation has special property of mapping solution curves of $\dot{q}_l = \alpha_l(q, \dot{q}, t)$ into curves $Q_l = Q_l(T)$ in Q, T space. If this transformation is to be a symmetry of Eq. (10) then, $Q_l = Q_l(T)$ must be a solution of $\dot{Q}_l = \alpha_l(Q, \dot{Q}, T)$. To derive a criterion for this to be the case we assume that the transformation is associated with a Lie group.

To obtain a condition on the infinitesimal generator of the group let us suppose that Eq. (8) contain a parameter θ, with respect to which they form a Lie group. To first order in θ we may write as follows;

$$Q_l = q_l + \theta \eta_l(q, \dot{q}, t) \qquad T = t + \theta \xi(q, \dot{q}, t) \tag{12}$$

Differentiating where we get;

$$\frac{\partial Q_l}{\partial t} = \theta \left(\frac{\partial \eta_l}{\partial t} \right) \qquad \frac{\partial Q_l}{\partial q_m} = \delta_{lm} + \theta \left(\frac{\partial \eta_l}{\partial q_m} \right) \tag{13}$$

with similar expressions for the other derivatives appearing in Eq. (11). Using these results in Eq. (11) and expanding to first order in θ we obtain

$$\dot{Q}_l = \dot{q}_l + \theta(\dot{\eta}_l - \dot{\xi}\dot{q}_l), \tag{14}$$

where

$$\left.\begin{aligned} \dot{\eta}_l &= \frac{\partial \eta_l}{\partial t} + \left(\frac{\partial \eta_l}{\partial q_m} \right) \dot{q}_m + \left(\frac{\partial \eta_l}{\partial \dot{q}_m} \right) \alpha_m \\ \dot{\xi} &= \frac{\partial \xi}{\partial t} + \left(\frac{\partial \xi}{\partial q_m} \right) \dot{q}_m + \left(\frac{\partial \xi}{\partial \dot{q}_m} \right) \alpha_m \end{aligned}\right\} \tag{15}$$

From Eq. (12) and (14) we see that the infinitesimal generator of the group may be written in the form

$$E = \xi \left(\frac{\partial}{\partial t} \right) + \eta_l \left(\frac{\partial}{\partial q_l} \right) + \left(\dot{\eta}_l - \dot{\xi}\dot{q}_l \right) \left(\frac{\partial}{\partial \dot{q}_l} \right) \tag{16}$$

the infinite equations of transformation are

$$Q_l = \exp(\theta E) q_l \quad T = \exp(\theta E) t \quad \dot{Q}_l = \exp(\theta E) \dot{q}_l \tag{17}$$

Now the special form of the coefficient of $\frac{\partial}{\partial \dot{q}_l}$ in Eq. (16) guarantees that when the group acts upon a solution curve of Eq. (10), the result constitutes a curve in Q, T space, with the $\dot{Q}_l$ given by Eq. (17) being precisely the time derivatives along the curve. We now derive conditions on $\xi(q, \dot{q}, t)$ and $\eta_l(q, \dot{q}, t)$ which ensure that the resultant curve in Q, T space is itself a solution of $\ddot{Q}_l = \alpha_l(Q, \dot{Q}, T)$.

Suppose that the transformation Eq. (17) permutes solution of Eq. (10) among themselves, and let $\phi(q, \dot{q}, t)$ be a conserved quantity for Eq. (10). Then we may put $\exp(\theta E)(\phi(q, \dot{q}, t)) = \phi(\exp(\theta E)q, \exp(\theta E)\dot{q}, \exp(\theta E)t) = \phi(Q, \dot{Q}, T) = \Psi(\theta, q, \dot{q}, t)$

If $\phi(q, \dot{q}, t) = C_1$ for the solution $q_l = q_l(t)$, then $\phi(Q, \dot{Q}, T) = C_2$ for the solution $Q_l = Q_l(t)$; it then follows that $\Psi(\theta, q, \dot{q}, t)$ is also a conserved quantity. Again, in the expression of Ψ in powers of θ, each coefficient must itself be a conserved quantity. In particular, consideration of first order terms shows that $E\{\phi\}$ is a conserved quantity if ϕ is. Expressing this result in the form

$$\frac{d}{dt}E\{\phi\} = \frac{d}{dt}\left\{\xi\frac{\partial \phi}{\partial t} + \eta_l\frac{\partial \phi}{\partial q_l} + \left(\dot{\eta}_l - \dot{\xi}\dot{q}_l\right)\frac{\partial \phi}{\partial \dot{q}_l}\right\} = 0 \tag{18}$$

allows us to derive a condition which $\xi(q, \dot{q}, t)$ and $\eta_l(q, \dot{q}, t)$ must satisfy in order for Eq. (17) to generate a symmetry transformation of Eq. (10). This is done by explicitly carrying out the total time differentiation in Eq. (18), and simplifying the resultant expression by use of following relations;

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial q_l}\dot{q}_l + \frac{\partial \phi}{\partial \dot{q}_l}\alpha_l = 0 \tag{18a}$$

$$\frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{q}_l}\right) = -\frac{\partial \phi}{\partial q_l} - \frac{\partial \phi}{\partial \dot{q}_m}\frac{\partial \alpha_m}{\partial \dot{q}_l} \tag{18b}$$

$$\frac{d}{dt}\left(\frac{\partial \phi}{\partial q_l}\right) = -\frac{\partial \phi}{\partial \dot{q}_m}\frac{\partial \alpha_m}{\partial q_l} \tag{18c}$$

$$\frac{d}{dt}\left(\frac{\partial \phi}{\partial t}\right) = -\frac{\partial \phi}{\partial \dot{q}_m}\frac{\partial \alpha_m}{\partial t} \tag{18d}$$

Eq. (18a) is simply statement that ϕ is a conserved quantity. To derive Eq. (18b) we put

$$\frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{q}_l}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial \dot{q}_l}\right) + \frac{\partial}{\partial q_m}\left(\frac{\partial \phi}{\partial \dot{q}_l}\right)\dot{q}_m + \frac{\partial}{\partial \dot{q}_m}\left(\frac{\partial \phi}{\partial \dot{q}_l}\right)\alpha_m$$

$$= \frac{\partial}{\partial \dot{q}_l}\left[\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial q_m}\dot{q}_m + \frac{\partial \phi}{\partial \dot{q}_m}\alpha_m\right] - \frac{\partial \phi}{\partial q_l} - \frac{\partial \phi}{\partial \dot{q}_m}\frac{\partial \alpha_m}{\partial \dot{q}_l}$$

Because of Eq. (18a) the quantity in rectangular brackets vanishes, yielding Eq. (18b). Equations (18c) and (18d) may be obtained in a similar manner, and using Equations (18a), (18b), (18c), (18d) in the expanded form of Eq. (18) then yields;

$$\left(\frac{d}{dt}\right)E\{\phi\} = \left[\ddot{\eta}_l - \dot{q}_l\ddot{\xi} - 2\dot{\xi}\alpha_l - E\{\alpha_l\}\right]\left(\frac{\partial \phi}{\partial \dot{q}_l}\right) = 0$$

Since this must hold for any conserved quantity, we obtain the conditions

$$\ddot{\eta}_l - \dot{q}_l\ddot{\xi} - 2\dot{\xi}\alpha_l - E\{\alpha_l\} = 0 \qquad (l = 1,2,\ldots,p) \tag{19}$$

which constitute a set of p equations in the $p+1$ quantities

$\xi(q,\dot{q},t),\ \eta_l(q,\dot{q},t),\ldots,\eta_p(q,\dot{q},t)$.

This criterion is expressed in a more compact form than that given by Anderson *et al.* (1976), and its derivation does not require the use of the general Ovsjannikov theory (Ovsjannikov, 1969).

The transformation Eq. (17), with ξ and η_l subject to Eq. (19), can always be interpreted as being simply a point transformation of the $2p+1$ independent variables $q_l,\ \dot{q}_l,\ t$. In this case, the domain of the operators $\exp\{\theta E\}$ is the manifold of all possible values of the variables.

Example: As an example, we may consider the one dimensional harmonic oscillator, whose equation of motion, in suitably normalized coordinates may be written

$$\ddot{q} + q = 0 \tag{20}$$

so that $\alpha(q,\dot{q},t) = -q$ (21)

Then Eq. (19) becomes $\ddot{\eta} + \eta - \dot{q}\dddot{\xi} + 2q\dot{\xi} = 0$ (22)

If we assume the forms $\xi = \dot{q}\sigma(q,t),\ \eta = \eta(q,t)$, then Eq. (22) has the solutions

$$\xi = q\dot{q} \qquad \eta = -q^3 \tag{23a}$$

$$\xi = \dot{q}\sin t \qquad \eta = -q^2 \sin t \tag{23b}$$

$$\xi = \dot{q}\cos t \qquad \eta = -q^2 \cos t \tag{23c}$$

$$\xi = 0 \qquad \eta = q \tag{23d}$$

$$\xi = 0 \qquad \eta = \sin t \tag{23e}$$

$$\xi = 0 \qquad \eta = \cos t \tag{23f}$$

These symmetry transformations for the harmonic oscillator have been found previously by Anderson *et al.* (1976). We will later consider in more detail; the infinitesimal generator of the group is

$$E = q\dot{q}\left(\frac{\partial}{\partial t}\right) - q^3\left(\frac{\partial}{\partial q}\right) - \left(2q^2 + \dot{q}^2\right)\dot{q}\left(\frac{\partial}{\partial \dot{q}}\right) \tag{24}$$

Lagrangian Systems and Dynamical Symmetries

Now we will study the connection between dynamical symmetries and conserved quantities, limiting ourselves at first to systems derivable from a Lagrangian $L(q, \dot{q}, t)$. The general Neother- type conserved quantity (Neother, 1918b) is in the form

$$\Phi = \left(\xi\dot{q}_l - \eta_l\right)\left(\frac{\partial L}{\partial \dot{q}_l}\right) - \xi L + f \tag{25}$$

where $\xi = \xi(q, \dot{q}, t) \qquad \eta_l = \eta_l(q, \dot{q}, t) \qquad f = f(q, t)$ Now we could choose

$$f(q, \dot{q}, t) = -\left(\xi\dot{q}_l - \eta_l\right)\left(\frac{\partial L}{\partial \dot{q}_l}\right) + \xi L + \Psi(q, \dot{q}, t) \tag{26}$$

where $\Psi(q, \dot{q}, t)$ is any conserved quantity. In this case Eq. (25) becomes $\Phi = \Psi(q, \dot{q}, t)$

Therefore, any conserved quantity Ψ could be said to be associated with any arbitrary transformation defined by ξ, η_l . This degree of generality is clearly not profitable, so that f always assumed to be of the form $f(q, t)$

Now differentiating Eq. (25) totally with respect to time, we find

$$\dot{\Phi} = \left(\xi\dot{q}_l - \eta_l\right)F_1 - E\{L\} - \dot{\xi}L + \dot{f} \tag{27}$$

where $F_1 = \left(\frac{d}{dt}\right)\left(\frac{\partial L}{\partial \dot{q}_l}\right) - \frac{\partial L}{\partial q_l}$

If Φ is conserved so that $\dot{\Phi} = 0$, and if the Euler equations are satisfied, so that $F_1 = 0$, then

$$E\{L\} + \dot{\xi}L = \dot{f}. \tag{28}$$

Conversely, if an $f(q, t)$ can be found such that Eq. (28) is satisfied, then Φ is conserved.

Now referring to the Eq. (19) for the existence of dynamical symmetry, we note that if $\xi = \xi(q, \dot{q}, t)$ is chosen arbitrarily, the equation may be solved for $\eta_1(q, \dot{q}, t), \ldots, \eta_p(q, \dot{q}, t)$. Because of this freedom in the choice of $\xi(q, \dot{q}, t)$, we can construct an infinity of independent dynamical symmetries, all of which fail to satisfy Eq. (28).Thus none of this constructed symmetries will determine a Neother quantity Eq. (25). Similarly, in Eq. (28) we may arbitrarily assign the $p+1$ functions $f(q, t), \eta_l(q, \dot{q}, t)$,and solve the resulting differential equation for $\xi(q, \dot{q}, t)$. Each of these sets ξ, η_l , then determines a conserved quantity Φ ; and because the η_l are arbitrary, the sets can be so determined that do not satisfy Eq.

(19). Now we may therefore construct a $p+1-$fold infinity of conserved quantities, each having the property that the associated ξ, η_l do not define a dynamical symmetry. Therefore, in this approach, the Lagrangian formalism is not utilized, and the particular representation of a conserved quantity in Neother form is not available.

General Systems and Dynamical Symmetries

Suppose that the general solution of Eq. (10) may be written as

$$q_l = q_l(t) \qquad (l = 1, 2, \ldots, p) \tag{29}$$

and depends on $2p$ arbitrary constants A_k. A set of $2p$ equations may be obtained by joining to Eq. (29) the p equations obtainable by time differentiation; this resulting set determines the $2p$ constants as functions of $q_l, \dot{q}_l$ and t. These functions of $q_l, \dot{q}_l$ and t are of course constant in time, since the A_k are constants; moreover, any combination of these functions yields further conserved quantities. Naturally, this procedure is not of much use in determining conserved quantities, since it requires that the general solution already be known. If a dynamical symmetry is known, then the above considerations lead to an approach which can yield a conserved quantity without knowledge of general solution.

Any transformation which carries one solution of Eq. (10) into another can be considered to have the effect of changing one set of constants $A_1, A_2, \ldots, A_{2p}$ into another set $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{2p}$. In particular, if a one-parameter Lie group permutes solutions among themselves, then there exists a corresponding Lie group acting on the constants $A_1, A_2, \ldots, A_{2p}$. Suppose that this group has an invariant $S(A_1, A_2, \ldots, A_{2p})$; that is, a function of the A_k having the property

$$S(A_1, A_2, \ldots, A_{2p}) = S(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{2p}) \tag{30}$$

From this invariant can be constructed a conserved quantity $I(q, \dot{q}, t)$, by using the above- mentioned expression for the A_k in terms of $q_l, \dot{q}_l$ and t. Specifically, if we replace the constants by their representations in terms of coordinates, velocities and time, then Eq. (30) assumes the form;

$$I(q, \dot{q}, t) = I(Q, \dot{Q}, T) \tag{31}$$

and thus $I(q, \dot{q}, t)$ is a conserved quantity for Eq. (10). It is natural to consider this constant of the motion $I(q, \dot{q}, t)$ as being associated with the given dynamical symmetry. Furthermore, $I(q, \dot{q}, t)$ can be found without knowing the general solution of Eq. (10), since it is clear from the relationship between $I(q, \dot{q}, t)$ and $S(A_1, A_2, \ldots, A_{2p})$ that $I(q, \dot{q}, t)$ is an invariant of the dynamical symmetry group itself; that is,

$$E\{I\} = 0 \tag{32}$$

To show this analytically we may apply the group operation Eq. (17) to the invariant $I(q, \dot{q}, t)$ and obtain

$$\exp(\theta E) I(q, \dot{q}, t) = I(Q, \dot{Q}, T) = S(\tilde{A}) = S(A) = I(q, \dot{q}, t) \tag{33}$$

From Eq. (33) we note that $\exp(\theta E) I(q,\dot{q},t) = I(q,\dot{q},t)$, form which Eq. (32) follows. This result leads to a method whereby a conserved quantity for Eq. (10) might be sought when a dynamical symmetry of Eq. (10) is known. The procedure required that the invariants of the symmetry group be found, for instance by solution of the invariants of the symmetry group be found, for instance by solution of Eq. (32). Among these invariants will be found the conserved quantity $I(q,\dot{q},t)$.

The considerations leading to Eq. (32) may be applied also to point symmetries if we interpret the operator Eq. (16) as representing the generator of the first extension of the group

$G = \xi\left(\frac{\partial}{\partial t}\right) + \eta_l(q_l)$; here ξ and η_l are independent of $\dot{q}$, and

$$\dot{\xi} = \frac{\partial \xi}{\partial t} + \left(\frac{\partial \xi}{\partial q_l}\right)\dot{q}_l \qquad \dot{\eta}_l = \frac{\partial \eta_l}{\partial t} + \left(\frac{\partial \eta_l}{\partial q_m}\right)\dot{q}_m$$

In particular, for Lagrangian systems we may now associate a conserved quantity with any point symmetry not possessing a Neother-type constant of motion; that is, with any point symmetry which leaves the equations of motion invariant but not the action (Lutzky, 1978).

Suppose the group generated by $E = \xi(q,t)\left(\frac{\partial}{\partial t}\right) + \eta_l(q,t)\left(\frac{\partial}{\partial q_l}\right)$ be a point symmetry of a dynamical system. Then a conserved quantity for the system may be associated with this symmetry group, and may be found among the invariants of the first extended group.

The fact that Eq. (32) holds for a dynamical symmetries enables us to state a theorem for dynamical symmetries which has previously been proved only for point symmetries; if a dynamical symmetry changes a given solution into another, then both solutions possess the same value of the conserved quantity associated with the symmetry. This follows form Eq. (32) because $E\{I\}$ gives the change in invariant *I* due to the change from one solution to another brought about by the action of the symmetry operator.

Discussion

Neother (1882-1935), one of the leading mathematicians of this century, has been properly described as 'the greatest of the women mathematicians'. The original publication of the theorem was in the Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, pp. 200-235 (1918a). In this study, for closed systems all the conservation laws due to homogeneity of space and time and isotropy of space are derived by using Neother's celebrated theorem (Neother, 1918a).

Elaborate discussions of symmetry properties (Gibson and Pollard, 1980; Emmerson, 1972) are studied and also we have studied Lie group (Lie, 1967; Greiner and Müller, 1994) which are required in connection with commutations relations. Symmetries of the laws of nature that are related to the conservation are derived analytically from its related symmetry for a simple mechanical system.

Properties of dynamical symmetries, Lagrangian systems and General systems are described briefly. The connection between conserved quantities and dynamical symmetries in the case of Lagrangian systems and dynamical symmetries and conserved quantities for general dynamical systems are discussed to reach the goal.

Conclusion

To conclude we must bear in mind that the vast knowledge of the micro world is knocking at the door. The beautiful feature of the world symmetry is very fascinating. Fresh researchers like us start just from footsteps of great scientists. Various properties of dynamical symmetries are derived. The connection between conserved quantities and dynamical symmetries in the case of Lagrangian systems are derived. In this approach we have reached a condition that the Lagrangian formalism is not utilized and the particular representation of a conserved quantity in Neother form is not available. Also, the connection between dynamical symmetries and conserved quantities for general dynamical systems are derived. In this approach we conclude that the conserved quantities associated with a given dynamical symmetries are invariants of the symmetry group itself and the conserved quantities associated with a point symmetry group are invariants of the first extended group. Above all, we believe that this study will play pioneer role in exploring further study in the field of Mathematical physics.

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