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SUPERIORITY OF WAVELET THEORY COMPARED TO FOURIER TRANSFORM

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Abstract: Wavelet analysis is an exciting new method for solving difficult problems in mathematics, physics, engineering and medical sciences. Signal transmission is based on transmission of a series of numbers. The series representation of a function is important in all types of signal transmission. The wavelet representation of a function is a new technique. We can say that wavelet transform of a function is the improved version of Fourier transform. In this study, the classical Fourier analysis is compared with the wavelet tools.

Key words: Fourier transforms, wavelet, wavelet transform, time-frequency analysis

Introduction

In 1982 Jean Morlet a French geophysicist, introduced the concept of a 'wavelet'. The wavelet means small wave and the study of wavelet transform is a new tool for seismic signal analysis. Immediately, Alex Grossmann theoretical physicists studied inverse formula for the wavelet transform. The joint collaboration of Morlet and Grossmann (Grossmann *et al*., 1984) yielded a detailed mathematical study of the continuous wavelet transforms and their various applications, of course without the realization that similar results had already been obtained in 1950's by Calderon, Littlewood, Paley and Franklin. However the rediscovery of the old concepts provided a new method for decomposing a function or a signal. For details one can see Morlet *et al.* (1982a, 1982b), Debnath (2002), Siddiqi (2004).

Wavelet analysis is originally introduced in order to improve seismic signal analysis by switching from short-time Fourier analysis to new better algorithms to detect and analyze abrupt changes in signals Daubechies (1988, 1990, 1992), Mallat (1998). In time-frequency analysis of a signal, the classical Fourier transform analysis is inadequate because Fourier transform of a signal does not contain any local information. This is the major drawback of the Fourier transform. To overcome this drawback, Dennis Gabor in 1946, first introduced the windowed-Fourier transform, i.e. short-time Fourier transform known later as Gabor transform. He also used Gaussian distribution function as the window function.

Meyer (1985) found the existing literature of wavelets. Later many eminent mathematicians e.g. I. Daubechies, A. Grossmann, S. Mallat, Y. Meyer, R. A. deVore, Coifman, V. Wickerhauser made a remarkable contribution to the wavelet theory. The modern applications of wavelet theory as diverse as wave

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propagation, data compression, image processing, pattern recognition, computer graphics, the detection of aircraft and submarines, improvement in CAT scans, some medical image technology etc.

Definition of Wavelet

The wavelet means small waves and in brief, a wavelet is an oscillation that decays quickly. Equivalent mathematical conditions for wavelet are:

$$\int_R |\psi(t)|^2 dt < \infty \quad \text{....(1)}$$

$$\int_R \psi(t) dt = 0 \quad \text{.....(2)}$$

$$\int_R \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \quad \text{.....(3)}$$

where $\hat{\psi}(t)$ is the Fourier Transform of $\psi(t)$.

Wavelet Transforms

Jean Morlet in 1982 introduced the idea of the wavelet transform, providing a new mathematical tool for seismic wave analysis. Morlet first considered the idea of wavelets as a family of functions constructed from translations and dilations of a signal function called the "mother wavelet" $\psi(t)$. They are defined by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in R, \ a \neq 0 \quad \text{.....(4)}$$

where a is the scaling parameter which measures the degree of compression or scale, and b is the translation parameter which determines the time location of the wavelet. If $|a|<1$, the wavelet in (4) is the compressed version (smaller support in time- domain) of the mother wavelet and corresponds mainly to higher frequencies. On the other hand, when $|a|>1$, then $\psi_{a,b}(t)$ has a larger time-width than $\psi(t)$ and corresponds to lower frequencies. Thus, wavelets have time-widths adapted to their frequencies. This is the main reason for the success of the Morlet wavelets in signal processing and time-frequency analysis Morlet *et al*. (1982a, 1982b).

Continuous Wavelet Transform

In wavelet theory, a function is represented by infinite series expansion in terms of dilated and translated version of a basis function. Wavelet transform of $f \in L_2(R)$ is defined by

$$\begin{aligned} T_\psi f(a,b) &= |a|^{-1/2} \int_{-\infty}^{+\infty} f(t) \psi\left(\frac{t-b}{a}\right) dt \\ &= \langle f, \psi_{a,b} \rangle \end{aligned} \quad \text{.....(5)}$$

where $\psi_{a,b}(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right)$ is translated by b and dilated by a of ψ and $T_\psi f(a,b)$ is called the wavelet transform of $f(t)$ in $L_2(R)$.

Note that $\psi_{a,b}(t)$ plays the same role as the kernel $e^{i\omega t}$ in the Fourier transform. Like the Fourier transform, the continuous wavelet transform $T_\psi f(a,b)$ is linear. However, unlike the Fourier transform, the continuous wavelet transform is not a single transform. The inverse wavelet transform can be defined so that $f(t)$ can be reconstructed by the formula

$$f(t) = C_{\psi}^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_{\psi} f(a,b) \psi_{a,b}(t)(a^{-2}da)db \qquad .....(6)$$

provided C_{ψ} satisfies the admissibility condition $C_{\psi} = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty,$ where $\hat{\psi}(\omega)$ is the Fourier Transform of ψ (t).

Discrete Wavelet Transform

The continuous wavelet transform can be computed at discrete grid points (a_n, b_n), $n \in Z$. To do this, a general wavelet ψ can be defined as

$$\psi_{m,n}(t) = a_0^{m/2} \psi(a_0^{m} t - b_0 n), \; m,n \in Z, \qquad .....(7)$$

where $a_0 > 1$ and $b_0 > 1$ are fixed parameters. For such a family, two vital questions arise.

i) Does the sequence $\{\langle f, \psi_{m,n} \rangle\}_{m,n \in Z}$ completely characterize the function f ?

ii) Is it possible to recover f from this sequence in a stable manner?

Classical Fourier Transform and its Drawback

Fourier series is a useful tool for representation and approximation of periodic functions through trigonometric functions. Fourier analysis is a mathematical technique for transforming our view of the signal from time-based to frequency-based. In transforming to the frequency domain, time information is lost. When looking at a Fourier transform of a signal, it is impossible to tell when a particular event took place.

In Fourier analysis signal properties do not change over time; that is if it is what is called a stationary signal. This drawback is not very important. But most interesting signals contain numerous non-stationary or transitory characteristics like drift, trends, abrupt changes and beginnings and ends of event. These characteristics are often the most important part of the signal. The classical Fourier analysis is not suited for detecting them.

Why Wavelet Analysis?

All types of signal transmission are based on transmission of a series of numbers. For signal transmission or signal storage the first step is to convert the given information to a series of numbers. To do this we need to represent a function f as a series representation. The function f is stored in the coefficients of the series and we can send only the coefficients. In practice we cannot send an infinite sequence of numbers. It is possible to send only a finite sequence of numbers. For good approximation usually this number forces to be large.

For series representation of a function, we consider a given function or signal f as

$$f(x) = \sum_{n=0}^{\infty} a_n f_n(x) \qquad .....(8)$$

where a_n 's are constant coefficients and $f_0, f_1, f_2, ...$ are simple functions.

The simple functions f_n may be polynomials or trigonometric functions.

For Taylor series approximation f should be analytic function.

For Fourier series approximation f should be periodic function. For details one can see Walnut (2001).

Ex. In signal analysis it is common to consider a function in $L_2(R)$.

i.e. $L_2(R) = \left\{ f : R \to C / \int_R |f(x)|^2 < \infty \right\}$. This is never periodic except f = 0. In that case we consider a wavelet function ψ such that

$$f(x) = \sum_{j \in Z} \sum_{k \in Z} d_{j,k} \psi_{j,k}(x) \qquad .....(9)$$

where $d_{j,k}$ are wavelet coefficients and $\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$ are the translated and scaled version of wavelet ψ.

For a periodic function the classical method is Fourier Transform. But the main drawback of Fourier Transform is that we lose our time information, which is very important. The wavelet tool is the new tool to represent a function like (8). In the wavelet transform we do not lose the time information. It is useful in many contexts. The advantage of wavelet theory includes-(i) it functions are well localized; (ii) it is often possible to obtain a good approximation of the given function *f* by using only a few coefficients; (iii) most of the wavelet coefficients $\{d_{j,k}\}_{|j|,|k| \geq N}$ vanish for large *N*.; (iv) it is capable of revealing aspects of data that other signal analysis techniques miss the aspects like trends, breakdown points, and discontinuities in higher derivatives and self-similarity; and, (v) it can often compress or de-noise a signal without appreciable degradation.

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