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RESEARCH ARTICLE PHYSICAL SCIENCE

AN EASIER ASYMPTOTIC METHOD FOR SOLVING SECOND ORDER OVER-DAMPED NONLINEAR SYSTEMS

M. Samsuzzoha[*], M. Golam Azom, B.K. Mondal and M.S. Anis

Mathematics Discipline, Khulna University, Khulna 9208, Bangladesh

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Abstract: By means of the extended Krylov-Bogoliubov-Mitropolskii method, an asymptotic solution of second order over-damped nonlinear system is found. The results obtained by this method are exactly same as the results obtained by Murty *et al.* (1969). The determination of the solution followed by Murty *et al.* (1969) is too much laborious and cumbersome. On the contrary, the present method is very simple and easier. It is illustrated by an example.

Key words: Asymptotic method, second order, nonlinear, Krylov-Bogoliubov-Mitropolskii method

Introduction

Among the methods used to study nonlinear systems with a small non-linearity, Krylov and Bogoliubov (1947) and Bogoliubov and Mitropolskii (1961) (KBM) method is particularly convenient and is the widely used technique to obtain the approximate solution. Originally, the method developed for systems with periodic solutions, and later extended by Popov (1956) and Meldelson (1970) for damped nonlinear oscillations. Murty *et al.* (1969) extended the method to over-damped nonlinear systems. Murty (1971) has also developed a unified KBM method for solving second order nonlinear systems. Sattar (1993) has studied a third order over-damped nonlinear system. Alam and Sattar (1996) developed a method for third order critically damped nonlinear equations. Alam and Sattar (1997) have also presented a unified KBM method for solving third order nonlinear systems. Recently, Alam (2002) has presented a unified method for solving an n-th order nonlinear differential equation and a perturbation theory Alam (2002) with large damping. However, the method presented in Murty *et al.* (1969) is too much laborious and cumbersome.

In this article a solution of a second order over-damped system is found based on the work of Alam (2002). The determination of the solution is very simple and easier than Murty *et al.* (1969). An example is solved in order to illustrate the method. The results obtained by the present method compare very well with those obtained by the numerical method and is exactly same as those the results obtained by Murty *et al.* (1969).

Materials and Methods

Consider a weakly nonlinear over-damped system governed by the differential equation

$$\ddot{x} + k_1 \dot{x} + k_2 x = \varepsilon f(x) \qquad (1)$$

where over-dots denote differentiation with respect to t, ε is a small parameter, $f(x)$ is the given nonlinear function and k_1, k_2 are constants defined by $k_1 = \lambda_1 + \lambda_2$, and $k_2 = \lambda_1 \lambda_2$.

[*]Corresponding author. Cell: 0156-319100; e-mail: <zohamathku@yahoo.com>

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It is noted that equation (1) is like to the equation of Murty *et al.* (1969). Here λ_1 and λ_2 are the real negative eigen-values of the characteristic equation of the unperturbed equation (1). The over damping force in the system is represented by these real negative eigen-values. Therefore, the unperturbed solution is

$$x(t,0) = a_{1,0}e^{\lambda_1 t} + a_{2,0}e^{\lambda_2 t} \tag{2}$$

where $a_{j,0}$, $j = 1, 2$ are arbitrary constants.

When $\varepsilon \neq 0$, following Alam (2002) the solution of the nonlinear differential equation (1) is sought in the form $x(t,\varepsilon) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \varepsilon u_1(a_1, a_2, t) + \varepsilon^2 u_2(a_1, a_2, t) + \varepsilon^3 \ldots$ (3)

where each a_j, $j = 1, 2$ satisfies the differential equation

$$\dot{a}_j(t) = \varepsilon A_j(a_1, a_2, t) + \varepsilon^2 B_j(a_1, a_2, t) + \varepsilon^3 \ldots, \tag{4}$$

Confining only to the first few terms, 1, 2, 3, …,*m* in the series expansions of (3) and (4), we evaluate the functions $u_1, u_2, \ldots$ and $A_j, B_j, j = 1, 2$, such that $a_j(t)$, appearing in (3) and (4) satisfy the given differential equation (1) with an accuracy of ε^{m+1}. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first Murty (1971). In order to determine these functions it is assumed that the functions $u_1, u_2, \ldots$ do not contain the fundamental terms, which are included in the series expansion (3) at order ε^0.

Differentiating (3) twice with respect to t, substituting *x* and the derivatives $\dot{x}, \ddot{x}$ in the original equation (1), utilizing the relations in (4) and equating the coefficients of ε, we obtain

$$\left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right)u_1 + e^{\lambda_1 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)A_1 + e^{\lambda_2 t}\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_1\right)A_2 = f^{(0)}(a_1, a_2, t), \tag{5}$$

where $f^{(0)} = f(x_0)$ and $x_0 = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$.

In general, the function $f^{(0)}$ can be expanded in a Taylor series as

$$f^{(0)} = \sum_{m_1=-\infty, m_2=-\infty}^{\infty,\infty} F_{m_1,m_2} a_1^{m_1} a_2^{m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t}.$$

According to our assumption u_1 does not contain the fundamental terms, therefore equation (5) can be separated into three equations for unknown functions u_1 and A_1, A_2 (see Alam (2002) for details). Substituting the functional values of $f^{(0)}$ and equating the coefficients of $e^{\lambda_j t}$, $j = 1, 2,$ we obtain

$$\begin{aligned} & e^{\lambda_1 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)A_1 \\ & = F_{m_1,m_2} a_1^{m_1} a_2^{m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t}; \qquad m_1 = m_2 + 1 \end{aligned} \tag{6}$$

$$e^{\lambda_2 t}\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_1\right)A_2$$
$$=F_{m_1,m_2}a_1^{\,m_1}a_2^{\,m_2}e^{(m_1\lambda_1+m_2\lambda_2)t};\qquad m_1=m_2-1 \tag{7}$$

and

$$\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_1$$
$$=\sum_{m_1=-\infty,m_2=-\infty}^{\infty,\infty}{}^{/}\;F_{m_1,m_2}a_1^{\,m_1}a_2^{\,m_2}e^{(m_1\lambda_1+m_2\lambda_2)t} \tag{8}$$

where $\sum^{/}$ exclude those terms for, $m_1=m_2\pm1$.

The particular solutions of the equations (6)-(8) give the functions A_1, A_2 and u_1. Thus the determination of the first approximate solution is complete.

Example

As an example of the method consider the second order differential equation

$$\ddot{x}+k_1\dot{x}+k_2x=\varepsilon x^3 \tag{9}$$

For equation (9), we have, $f^{(0)}=(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t})^3$. or,

$$f^{(0)}=a_1^{\,3}e^{3\lambda_1 t}+a_2^{\,3}e^{3\lambda_2 t}+3a_1^{\,2}a_2e^{(2\lambda_1+\lambda_2)t}+3a_2^{\,2}a_1e^{(2\lambda_2+\lambda_1)t}. \tag{10}$$

Therefore, the equations (6)-(8) become

$$e^{\lambda_1 t}\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)A_1=3a_1^{\,2}a_2e^{(2\lambda_1+\lambda_2)t} \tag{11}$$

$$e^{\lambda_2 t}\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_1\right)A_2=3a_2^{\,2}a_1e^{(2\lambda_2+\lambda_1)t} \tag{12}$$

and

$$\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_1=a_1^{\,3}e^{3\lambda_1 t}+a_2^{\,3}e^{3\lambda_2 t}. \tag{13}$$

Solving the equations (11)-(13) and substituting, $\lambda_1=-k+\omega$ and $\lambda_2=-k-\omega,$ we obtain

$$A_1 = \frac{3a_1^{\,2} a_2 e^{-2kt}}{-2(k-\omega)},$$
$$A_2 = \frac{3a_1 a_2^{\,2} e^{-2kt}}{-2(k+\omega)} \qquad (14)$$

Substituting the values of (14) into equation (4), we obtain

$$\dot a_1 = -\frac{3\varepsilon a_1^{\,2} a_2 e^{-2kt}}{2(k-\omega)},$$
$$\dot a_2 = -\frac{3\varepsilon\, a_1 a_2^{\,2} e^{-2kt}}{2(k+\omega)} \qquad (15)$$

Substituting $a_1 = \dfrac{a}{2}e^{\varphi}$ and $a_2 = \dfrac{a}{2}e^{-\varphi}$ into equation (15) and then simplifying, we obtain

$$\dot a = \varepsilon\, l_1 k\, a^3 e^{-2kt},$$
$$\dot\varphi = \varepsilon\, l_1 \omega\, a^2 e^{-2kt} \qquad (16)$$

where, $l_1 = -\dfrac{3}{8(k^2-\omega^2)}$. and,

$$u_1 = a^3\{m_1 \cosh 3(\omega t+\varphi) + m_2 \sinh 3(\omega t+\varphi)\}e^{-3kt} \qquad (17)$$

where, $m_1 = \dfrac{k^2+2\omega^2}{16(k^2-\omega^2)(k^2-4\omega^2)}$ and, $m_2 = \dfrac{3k\omega}{16(k^2-\omega^2)(k^2-4\omega^2)}$

Solving equation (16), we obtain

$$a = a_0 \Big/ \sqrt{1+\varepsilon\, l_1 a_0^{\,2}(e^{-2kt}-1)}$$
$$\varphi = \varphi_0 - \frac{\omega}{2k}\log\left\{\sqrt{1+\varepsilon l_1 a_0^{\,2}(e^{-2kt}-1)}\right\} \qquad (18)$$

Therefore, we obtain the first order approximate solution of equation (9) as

$$x = ae^{-kt}\cosh(\omega t+\varphi) + \varepsilon u_1, \qquad (19)$$

where a and φ are given by (18) and u_1 is given by (17).

Murty, Deekshatulu and Krishna's (1969) Solution

Murty *et al.* (1969) considered a second order differential equation as-

$$\ddot{x} + 2K_1\dot{x} + K_{11}x = \varepsilon f(x) \tag{20}$$

which is similar ($k_1 = 2K_1,\, k_2 = K_{11}$) to the equation (1) and they found a solution of the equation (20) in the form - $x(t) = a(t)\cosh\psi(t) + \varepsilon u_1(a,\psi) + \ldots,$ (21)

where $u_i(a,\psi),\quad i = 1, 2, 3, \ldots$ are functions of a and ψ, and the quantities a and ψ are defined by the differential equations -

$$\frac{da}{dt} = -K_1 a + \varepsilon A_1(a) + \ldots,\ \frac{d\psi}{dt} = -K_2 + \varepsilon B_1(a) + \ldots, \tag{22}$$

and

$$K_1 = \frac{\lambda_1 + \lambda_2}{2},\ K_2 = \frac{\lambda_1 - \lambda_2}{2}.$$

Differentiating the equation (21) twice with respect to t, using the relations in (22), substituting x and the derivatives $\dot{x}$ and $\ddot{x}$ in the equation (20), comparing the coefficients of equal power of ε, Murty *et al.* (1969) obtained

$$\begin{aligned}
&K_1^2\left(a^2\frac{\partial^2 u_1}{\partial a^2} - a\frac{\partial u_1}{\partial a}\right) + K_2^2\frac{\partial^2 u_1}{\partial\psi^2} + 2K_1K_2\left(a\frac{\partial^2 u_1}{\partial a\partial\psi} - \frac{\partial u_1}{\partial\psi}\right)\\
&+ K_{11}u_1 - \cosh\psi\left[K_1\left(-A_1 + a\frac{dA_1}{da}\right) + 2K_2 aB_1\right]\\
&- \sinh\psi\left[2K_2A_1 + -K_1 a\frac{dB_1}{da}\right] = f(x)
\end{aligned} \tag{23}$$

Expanding the right hand side into Taylor's series, substitute for x from equation (21), express the powers of $\cosh\psi$ in terms of multiple arguments of $\cosh\psi$ and equating the coefficients of simple and higher arguments of $\cosh\psi$ and $\sinh\psi$ in the resulting equation, they obtained

$$K_1\left(-A_1 + a\frac{dA_1}{da}\right) + 2K_2 aB_1 = h_1(a) \ \text{for} \ \cosh\psi$$

$$2K_2A_1 + -K_1 a\frac{dB_1}{da} = g_1(a) \quad \text{for} \ \sinh\psi \tag{24}$$

and for higher arguments of $\cosh\psi$ and $\sinh\psi$

$$K_1^2\left(a^2\frac{\partial^2 u_1}{\partial a^2}-a\frac{\partial u_1}{\partial a}\right)+K_2^2\frac{\partial^2 u_1}{\partial \psi^2}+2K_1K_2\left(a\frac{\partial^2 u_1}{\partial a\partial\psi}-\frac{\partial u_1}{\partial \psi}\right)+K_{11}u_1$$
$$=\sum_{r=2}^{\infty}\left(h_r(a)\cosh r\psi+g_r(a)\sinh r\psi\right) \qquad (25)$$

where the functions h_1, g_1, h_r and g_r are the coefficients of the fundamental and higher argument terms in $\cosh\psi$ and $\sinh\psi$, respectively, obtained in the Taylor's series expansion.

Elimination of A_1 from the simultaneous equation (24) results in Euler type equation in B_1 whose particular solution gives B_1 and hence A_1. Substituting the values of A_1 and B_1 into the equation (22) and integrating Murty *et al.* (1969) obtained the values of *a* and ψ and thus the determination of the solution was completed.

As an example they considered

$$\ddot{x}+3\dot{x}+2x=\varepsilon x^3 \qquad (26)$$

Here, $2K_1=3,\quad K_{11}=2$ and $f(x)=x^3$.

Therefore, equation (24) becomes

$$K_1\left(-A_1+a\frac{dA_1}{da}\right)+2K_2aB_1=-\frac{3a^2}{4}\ \text{for}\ \cosh\psi$$

$$2K_2A_1+-K_1a\frac{dB_1}{da}=0\quad \text{for}\ \sinh\psi \qquad (27)$$

Therefore, the particular solution of A_1 and B_1 are

$$A_1=\frac{3K_1}{8(K_2^2-K_1^2)}a^3$$
$$B_1=\frac{-3K_2}{8(K_2^2-K_1^2)}a^2 \qquad (28)$$

Substituting the values of A_1 and B_1 in equation (22), they obtained

$$\frac{da}{dt}=-K_1a+\varepsilon\frac{3K_1}{8(K_2^2-K_1^2)}a^3+\ldots,$$
$$\frac{d\psi}{dt}=-K_2-\varepsilon\frac{3K_2}{8(K_2^2-K_1^2)}a^2+\ldots \qquad (29)$$

Solutions of the equation (29) are

$$a = \frac{a_0 e^{-K_1 t}}{\{1 + \frac{3\varepsilon\, a_0^2}{8(K_2^2 - K_1^2)}(e^{-2K_1 t} - 1)\}^{\frac{1}{2}}}$$

$$\psi = \psi_0 - K_2 t + \frac{K_2}{2K_1} \log\{1 + \frac{3\varepsilon\, a_0^2}{8(K_2^2 - K_1^2)}(e^{-2K_1 t} - 1)\} \tag{30}$$

and the solution of the equation (25) is

$$u_1 = \frac{12K_1K_2 \sinh 3\psi - (3K_1^2 + K_{11} + 9K_2^2)\cosh 3\psi}{4\{144K_1^2K_2^2 - (3K_1^2 + K_{11} + 9K_2^2)^2\}} a^3 \tag{31}$$

Therefore, the solution of the equation (26) is

$$x(t) = a(t)\cosh\psi(t) + \varepsilon u_1 \tag{32}$$

where a, ψ and u_1 are given in (30) and (31) respectively.

Results

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we sometimes compare the approximate solution to the numerical solution.

Our approximate solution x , evaluated from (19) with initial conditions $a_0 = 2,\ \varphi(0) = 0.5,\ \varepsilon = 0.1$ for various values of *t*, is presented in the second column of the Table 1. Corresponding numerical solutions computed by a fourth order Runge-Kutta method is designed by x_1, in the third column of the Table 1. Computing the result of present study has been compared with the results x_{11} of Murty *et al.* (1969) and shown in Table 1.

Table 1. Comparison of our results with Murty, Deekshatulu and Krishna's (1969) values.

t	Present study x	Numerical study x_1	Murty *et al.* (1969) x_{11}
0.00	2.8888	2.8888	2.8888
0.30	1.3360	1.3360	1.3360
0.60	0.80851	0.80773	0.80851
0.90	0.50293	0.50181	0.50293
1.20	0.32069	0.31953	0.32069
1.50	0.20929	0.20826	0.20929
1.80	0.13958	0.13872	0.13958
2.10	0.09494	0.09425	0.09494
2.40	0.06569	0.06515	0.06569
2.70	0.04612	0.04570	0.04612

Discussion

From Table 1, we see that our asymptotic solution (19) and the solution presented in Murty *et al.* (1969) are exactly same. But the determination of this solution Murty *et al.* (1969) is too much laborious and cumbersome and tremendously complex with increasing the order of the differential equation, while the new technique is systematic, simple and easier than that of Murty *et al.* (1969).

Conclusion

An asymptotic method, based on the theory of Krylov-Bogoliubov-Mitropolskii, is presented in this article for the transient response of a nonlinear system governed by a second order ordinary differential equation, when the two characteristic roots of the corresponding linear equation are real and negative. The solution is presented as a power series in ε, where ε is a small parameter. The series itself is not convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as ε tends to zero. The results obtained by the present method compare very well with those obtained by the numerical method.

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