



A REACTION-ADVECTION-DIFFUSION MODEL IN SPATIAL ECOLOGY: THEORETICAL AND COMPUTATIONAL ANALYSIS

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Abstract

In a confined heterogeneous habitat with two species interacting for common resources, the research analyzes a reaction-advection-diffusion type dispersal model with homogeneous Neumann boundary conditions for generalized growth functions. Both species follow the same symmetric growths law, but their dispersal strategies and advection rates are different. The following pattern is used to consider the competition strategy: in a bounded heterogeneous habitat, the first population disperses according to its resource functions, whereas the second population disperses according to its carrying capacity. We investigate the model in two scenarios: when carrying capacity and resource functions are non-proportional, competitive exclusion occurs, and one species drives the other to extinction in the long run for various similar and unequal carrying capacities of competing species. However, coexistence is achievable for different resource distribution consumption if the resource distribution and the carrying capacity phase of the second species are non-constant and similar. A series of numerical computations are used to demonstrate the model's efficacy in one- and two-dimensional instances, which is particularly significant for environmental consideration.

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Introduction

The dispersion of populations in space and their preferred dispersal strategy in a varied environment is a significant and visible element in population biology. It is natural to wonder how different dispersal dynamics of those species result in patterns of geographical distribution? What types of patterns emerge from different processes, and why do organisms evolve to spread in specific ways? As well, which dispersal patterns can provide some sort of selection or ecological benefit? This is a crucial topic in the development of dispersal. To answer those questions, considerable effort has been made to use spatial models (see, Cantrell et al., 2008; Chen et al., 2012; Dockery et al., 1998; Korobenko and Braverman, 2009; Lam and Ni, 2012 and references therein).

There are numerous degrees of dispersal mechanisms for biological particles when using the diffusion model. If classical diffusion is the only way for species to disperse and there is no control, the relevant model is a diffusion equation with no reaction term. Where in some situations species can move to the area of poor available resources. This model has been thoroughly investigated theoretically and practically without the use of the advection effect (see, Brown, 1980; Cantrell and Cosner, 2003; de Mottoni, 1979; Dockery et al., 1998; Lam et al., 2020). Another method is the reaction-diffusion-advection model (see, Averill et al., 2012; Cantrell et al., 2007; Zhang et al., 2013) which is used when species detect an environmental gradient and migrate toward or away from it, which is more competitive than the prior gradient due to the advection term. Species, on the other hand, prefer to migrate through a favorable habitat and leave an undesirable one, this can cause erratic behavior (Geritz & Gyllenberg, 2008).

In the actual world, organisms tend to migrate to a better position for a variety of reasons, including food, safety, and other survival instincts. So, species don't just disperse at random, they choose to travel in search of a better life. Carrying capacity-driven diffusion (Braverman & Braverman, 2009) was introduced from this perspective. Species diffuse in this sort of dispersal technique, not to less fruitful locations, but in areas with higher per capita available resources. The solution obtained from K -based diffusion (carrying capacity-driven diffusion) coincides with the ideal free distribution, which represents a substantial change against the conventional dispersal strategy, and when compared to random diffusion, the K -driven diffusion method has a competitive advantage. The current version of resource-based diffusion was first introduced in (see, Braverman and Kamrujjaman, 2016a, 2016b; Kamrujjaman, 2019) where species disperse according to their dispersal distribution function and in this sort of diffusion and its solution corresponds with the K -driven ideal free distribution under particular conditions.

Our research is motivated in part by a desire to learn more about the development of dispersal in environments that are both geographically and temporally variable. In this context, we're looking at models for two rivals who utilize different dispersal methods but are otherwise ecologically equivalent, and we are looking at the strategies' evolutionary stability in terms of invasion. If the population shifts from areas with less accessible resources per capita to areas with more available resources per capita, we investigate the model when the diffusion approach isn't traditional (random) but rather guided, with the focus directed towards the highest abundance. This is the case in this work for two competing species for different resource functions with varying rates of advection, and an informative description can be found in the next section.

Let us consider the competition model for two competing species indicated by $u(t, x)$ and $v(t, x)$, respectively in a closed heterogeneous domain ω where they battle for analogues basis needs with homogeneous Neumann boundary conditions. Then the symmetric competition model is described as follows where their resource distributions are unequal:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot \left[a_1(x) \nabla \left(\frac{u(t, x)}{K_u(x)} \right) \right] + u(t, x) f \left(x, u(t, x), \frac{K_u(x)}{K_v(x)} v(t, x), K_u(x) \right), \quad t > 0, \quad x \in \omega, \\ \frac{\partial v}{\partial t} &= \nabla \cdot \left[a_2(x) \nabla \left(\frac{v(t, x)}{P(x)} \right) \right] + v(t, x) f \left(x, v(t, x), \frac{K_v(x)}{K_u(x)} u(t, x), K_v(x) \right), \quad t > 0, \quad x \in \omega, \\ \frac{\partial u}{\partial n} - \frac{u}{K_u} \frac{\partial K_u}{\partial n} &= \frac{\partial v}{\partial n} - \frac{v}{P} \frac{\partial P}{\partial n} = 0, \quad x \in \partial\omega, \\ u(0, x) &= u_0(x), \quad v(0, x) = v_0(x), \quad x \in \omega. \end{aligned} \tag{1}$$

Where $u(t, x), v(t, x) > 0$ are the density of populations and $K_u, K_v > 0$ are the competing species environmental carrying capabilities, respectively. Also, $P(x) > 0$ is the resource function which is smooth and positive. However, $K_u(x), K_v(x)$, and $P(x)$ are in the class $C^{1+\alpha}(\bar{\omega})$, for any $x \in \bar{\omega}$, where ω is a nonempty bounded open domain of \mathbb{R}^i with $\partial\omega \in C^{2+\alpha}$, $0 < \alpha < 1$. The set $h_1 \times h_2$ is a bounded subset of \mathbb{R}^2 that corresponds to the solution range to (1). Here $a_1(x)$ and $a_2(x)$ represent the spatial dependent rate of advection for u and v , respectively and we can express this advection-diffusion term as,

$$\begin{aligned} \nabla \cdot \left[a_1(x) \nabla \left(\frac{u(t, x)}{K_u(x)} \right) \right] &= \nabla \cdot \left[\left(\frac{a_1(x)}{K_u(x)} \right) \left(\nabla u(t, x) - u(t, x) \frac{\nabla K_u}{K_u} \right) \right], \\ \nabla \cdot \left[a_2(x) \nabla \left(\frac{v(t, x)}{P(x)} \right) \right] &= \nabla \cdot \left[\left(\frac{a_2(x)}{P(x)} \right) \left(\nabla v(t, x) - v(t, x) \frac{\nabla P}{P} \right) \right], \end{aligned}$$

respectively. Moreover, the boundary conditions in the system (1) are equivalent to

$$\frac{\partial(u/K_u)}{\partial n} = \frac{\partial(v/P)}{\partial n} = 0, \quad \text{for } x \in \partial\omega.$$

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The function $F(x, u, K)$ fulfills the subsequent properties for the presented growth functions, logistic (He & Ni, 2013), Gilpin-Ayala (Gilpin & Ayala, 1973), Nicholson's blowflies type growth (Gurney et al., 1980; Nicholson, 1954), food-limited (Smith, 1963) and Gompertz (Gompertz, 1825). Throughout the study we will use $g_1(x, u, v) = uf(x, u, v, K_u) = uf\left(x, u, \frac{K_u}{K_v}v, K_u\right)$ and $g_2(x, u, v) = vf(x, v, u, K_v) = vf\left(x, v, \frac{K_v}{K_u}u, K_v\right)$:

- (e1) $\mathbf{g} = (g_1, g_2)$ is quasi-monotone non-increasing in $h_1 \times h_2$;
- (e2) $g_i(\cdot, u, v)$ is Hölder continuous in ω and $g_i(x, \cdot, \cdot) \in C^2(h_1 \times h_2)$, $i = 1, 2$;
- (e3) $f(x, u_1, u_2, K) = f(x, u_2, u_1, K) = F(x, u_1 + u_2, K)$;
- (e4) $F(x, K, K) = 0$;
- (e5) $F(x, u, K)$ decreases in u in a strict monotone manner;
- (e6) $F(x, u, K)$ increases in K in a strict monotone manner;

Due to Allee effect (Kot, 2001) the property (e5) is not satisfied for the corresponding growth functions. For better understanding inaugurate the following change of variables $w(t, x) = u(t, x)/K_u(x)$, $p(t, x) = v(t, x)/P(x)$ for further analysis, then the system (1) becomes

$$\begin{aligned} K_u \frac{\partial w}{\partial t} &= \nabla \cdot (a_1 \nabla w) + K_u w f^*(x, w, K_u p / K_v, K_u), \quad t > 0, \quad x \in \omega, \\ K_v \frac{\partial p}{\partial t} &= \nabla \cdot (a_2 \nabla p) + K_v p f^*(x, p, K_v w / K_u, K_v), \quad t > 0, \quad x \in \omega, \\ \frac{\partial w}{\partial n} &= \frac{\partial p}{\partial n} = 0, \quad x \in \partial \omega. \end{aligned} \tag{2}$$

in which $f^*(x, w, p, K_u) = f(x, K_u w, Pp, K_u)$ and $f^*(x, p, w, K_v) = f(x, Pp, K_u w, K_v)$.

In this study the main findings are considered in two main folds:

1. Foremost, first species (u) following the K -driven advection-diffusion strategy where second one (v) moves towards smooth, positive resource function, and we see for unequal resource distribution species u which following the K -driven strategy will survive and another one is going to extinction.
2. The second situation is acquired while considering both are following the same K -driven advection-diffusion strategy and we will acquire coexistence for the system (1) for varied consumption rates.

3. However, the efficacy of the model (1) will be presented by numerical simulation both in one and two space dimensions cases which are more significant for ecological importance.

The following is how the paper is decorated. Firstly, we reported early results as well as the uniqueness and positivity of all solutions for any non-negative, non-trivial initial functions of the competition model (1). After this part, we look at the analysis of global equilibrium for both competitive exclusion and coexistence solutions. We will also present the study's main findings here. The numerical portion shows that all of the analytical results can be applied to a variety of growth laws, such as logistic growth laws. Finally, the study's synthesis is offered at the conclusion of the study.

1 Some Auxiliary Results and Positivity of Solution

We now describe the models which present the mechanism of single as well as a couple of species.

For the systems (1), it is simple to check that u^* and v^* are the steady state solutions to the preceding elliptic BVP (boundary value problems):

$$\nabla \cdot \left[a_1(x) \nabla \left(\frac{u^*(x)}{K_u(x)} \right) \right] + u^*(x) f(x, u^*(x), 0, K_u(x)) = 0, \quad x \in \omega, \quad \frac{\partial(u^*/K_u)}{\partial n} = 0, \quad x \in \partial\omega, \quad (3)$$

$$\nabla \cdot \left[a_2(x) \nabla \left(\frac{v^*(x)}{P(x)} \right) \right] + v^*(x) f(x, v^*(x), 0, K_v(x)) = 0, \quad x \in \omega, \quad \frac{\partial(v^*/P)}{\partial n} = 0, \quad x \in \partial\omega, \quad (4)$$

respectively. The upcoming Lemma follows from (Braverman and Braverman, 2009; Korobenko et al., 2013).

Lemma 1. (*Braverman & Braverman, 2009; Korobenko et al., 2013*) *For any non-negative, non-trivial initial values, $u^*(x) = K_u(x)$ is the only positive solution of (3). However, If $P(x) = K_v(x)$, then $v^*(x) = K_v(x)$ is the only positive solution of (4).*

The following next theorem exposes the existence and uniqueness of solutions for coupled systems (1).

Theorem 1. (*Korobenko & Braverman, 2014; Korobenko et al., 2013*) *Let the functions $g_1(x, u, v) = uf(x, u, K_u v/K_v, K_u)$ and $g_2(x, u, v) = vf(x, v, K_v u/K_u, K_v)$ satisfy (e1), (e2), and (e5). Then for any $u_0(x), v_0(x) \in C(\omega)$, system (1) has a unique solution $(u(t, x), v(t, x))$ so that $u(t, x) > 0, v(t, x) > 0$ for any $t > 0$.*

The proof can continue in same way following Theorem 1 in (Braverman et al., 2015) either for $P(x) \equiv K_v(x)$ or $P(x) \not\equiv K_v(x)$, therefore omitted it here.

Zahan, I. et al. (2022). A reaction-advection-diffusion model in spatial ecology: theoretical and computational analysis. *Khulna University Studies*, Special Issue (ICSTEM4IR): 203-218.

Lemma 2. Let $P(x)$, $K_u(x) \not\equiv K_v(x)$ are non-constant and the function f satisfy (e1)-(e6). If $P(x)/K_u(x) \not\equiv \text{constant}$ where $K_u(x) > K_v(x)$ in ω which is nonempty and open, then there sustains a unique positive solution $v^*(x)$ of (4) such that

$$\int_{\omega} f(x, 0, v^*, K_u) K_u(x) dx \equiv \int_{\omega} F(x, v^*, K_u) K_u dx > 0. \quad (5)$$

Proof. Integrating (4) over ω and imposing the homogeneous Neumann boundary conditions as well as applying (e3) for f , we get $\int_{\omega} v^*(x) F(x, v^*, K_v) dx = 0$.

Since by assumption, $K_u(x) > K_v(x)$ and from the property (e6), we have $F(x, v^*, K_u) > F(x, v^*, K_v)$, so

$$\int_{\omega} v^* F(x, v^*, K_u) dx > 0. \quad (6)$$

Integrating the inequality (6) and imposing the Mean Value Theorem along with property (e4)

$$v^* F(x, v^*, K_u) = (v^* - K_u)^2 F_2(x, \xi, K_u) + K_u F(x, v^*, K_u)$$

we get

$$\int_{\omega} K_u F(x, v^*, K_u) dx > - \int_{\omega} F_2(x, \xi, K_u) (v^* - K_u)^2 dx \quad (7)$$

where ξ lies in v^* and K_u . However, unless $v^* \equiv K_u \equiv \text{constant}$ the right-hand part is positive. Since due to (e5), $F_2 < 0$, which brings the proof to a conclusion. \square

Lemma 3. Let $P(x)$, $K_u(x) \not\equiv K_v(x)$ are non-constant and the function f satisfy (e1)-(e5). If $P(x)/K_u(x) \not\equiv \text{constant}$ and $K_u(x) = \nu K_v(x) \not\equiv \text{constant}$, where $\nu > 0$ on ω then the patronising inequalities are hold for (1)

$$\int_{\omega} f(x, v^*, 0, K_v) K_u dx \equiv \int_{\omega} F(x, v^*, K_v) K_u dx > 0, \quad (8)$$

and

$$\int_{\omega} f(x, v^*, 0, K_v) K_v dx \equiv \int_{\omega} F(x, v^*, K_v) K_v dx > 0. \quad (9)$$

Lemma 4. The trivial equilibrium $(0, 0)$ of the model (1) is unstable as well as repelling.

The proof is available in (Korobenko & Braverman, 2012, 2014), so we have omitted it here.

2 Analysis of Global Steady States

The stability of each steady state of the system (1) is discussed in the sections that follow: two semi-trivial equilibrium $(u^*, 0)$, $(0, v^*)$ when only single species survives, a coexistence equilibrium (u^*, v^*) in which both species mutually sustain in the same environment, and a trivial equilibrium $(0, 0)$ where both species depart the site in competition. Conditions (5), (8) and (9) will be applied throughout this study to address the global stability of the equilibrium points.

Lemma 5. *Let $P(x)$, $K_u(x) \not\equiv K_v(x)$ are non-constant and the growth function f satisfies (e1)-(e5). If $P(x)/K_u(x) \not\equiv$ constant and $K_u(x) = \nu K_v(x) \not\equiv$ constant, where $\nu > 0$ in a nonempty open domain, then no coexistence equilibrium (u_e, v_e) for system (1) is exist.*

Proof. Contrarily, imagine that a precisely positive equilibrium solution (u_e, v_e) of (1) occurs such that

$$\begin{aligned} \nabla \cdot \left[a_1 \nabla \left(\frac{u_e}{K_u} \right) \right] + u_e f(x, u_e, K_u v_e / K_v, K_u) &= 0, \quad x \in \omega, \quad \frac{\partial(u_e / K_u)}{\partial n} = 0, \quad x \in \partial\omega, \\ \nabla \cdot \left[a_2 \nabla \left(\frac{v_e}{P} \right) \right] + v_e f(x, v_e, K_v u_e / K_u, K_v) &= 0, \quad x \in \omega, \quad \frac{\partial(v_e / P)}{\partial n} = 0, \quad x \in \partial\omega. \end{aligned} \quad (10)$$

Multiplying the second equation in (10) by ν , adding the equations of u_e and v_e and applying (e3) for f , then integrating over ω as well as imposing the homogeneous boundary conditions we obtain for $K_u(x) = \nu K_v(x)$

$$\int_{\omega} (u_e + \nu v_e) F(x, u_e + \nu v_e, K_u) dx = 0. \quad (11)$$

Applying property (e6) for $K_u(x) > K_v(x)$, we find $\int_{\omega} F(x, u_e(x) + \nu v_e(x), K_u(x)) dx > \int_{\omega} F(x, u_e(x) + \nu v_e(x), K_v(x)) dx$.

Thus, $\int_{\omega} (u_e(x) + \nu v_e(x)) F(x, u_e(x) + \nu v_e(x), K_u(x)) dx > 0$ and integrating the inequality

$$(u_e + \nu v_e) F(x, u_e + \nu v_e, K_u) = F(x, u_e + \nu v_e, K_v)(u_e + \nu v_e - K_u) + K_u F(x, u_e + \nu v_e, K_u)$$

over ω , we obtain

$$0 < \int_{\omega} F(x, u_e + \nu v_e, K_u)(u_e + \nu v_e - K_u) dx + \int_{\omega} K_u(x) F(x, u_e + \nu v_e, K_u) dx. \quad (12)$$

The Mean Value Theorem for $F(x, K_u(x), K_u(x)) = 0$ and by (e4) gives

$$F(x, u_e(x) + \nu v_e(x), K_u(x)) = F_2(x, \xi, K_u(x))(u_e(x) + \nu v_e(x) - K_u(x)). \quad (13)$$

Zahan, I. et al. (2022). A reaction-advection-diffusion model in spatial ecology: theoretical and computational analysis. *Khulna University Studies*, Special Issue (ICSTEM4IR): 203-218.

where for $x \in \omega$ and for every $t > 0$, ξ lies in between $u_e(x) + \nu v_e(x)$ and $K_u(x)$. Using (13), inequality (12) can be rewritten as

$$\int_{\omega} K_u F(x, u_e + \nu v_e, K_u) dx > - \int_{\omega} F_2(x, \xi, K_u) (u_e + \nu v_e - K_u)^2 dx > 0. \quad (14)$$

the last inequality is positive where $F_2 < 0$ due to (e5) and which excludes the possibility $u_e + \nu v_e \equiv K_u$.

Now we consider the following issue $u_e + \nu v_e \not\equiv K_u$ and either $0 < \nu < 1$ or $\nu > 1$ for $K_u(x) \equiv \nu K_v(x)$ in ω . Consider the eigenvalue problem

$$\nabla \cdot \left[a_1 \nabla \left(\frac{\phi}{K_u} \right) \right] + \phi f(x, u_e + \nu v_e, K_u) = \sigma \phi, \quad x \in \omega, \quad \frac{\partial(\phi/K_u)}{\partial n} = 0, \quad x \in \partial\omega. \quad (15)$$

According to (Cantrell & Cosner, 2003), its principal eigenvalue is given by

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \frac{- \int_{\omega} a_1 |\nabla(\phi/K_u)|^2 dx + \int_{\omega} (\phi^2/K_u) f(x, u_e + \nu v_e, K_u) dx}{\int_{\omega} (\phi^2/K_u) dx}.$$

choosing $\phi = K_u$ and using the property (e3) for f , its principal eigenvalue is given by

$$\sigma_1 \geq \frac{\int_{\omega} K_u F(x, u_e + \nu v_s, K_u) dx}{\int_{\omega} K_u dx} \quad (16)$$

where $\sigma_1 > 0$ using (14). However, (w_e, p_e) is an equilibrium solution of (2), and hence a positive principal eigenfunction of (15) with the principal eigenvalue 0. This is in conflict with $\sigma_1 > 0$. This brings the proof to a conclusion. \square

Lemma 6. Let $P(x)$, $K_u(x) \not\equiv K_v(x)$ are non-constant and the growth function f satisfies (e1)-(e5). If $P(x)/K_u(x) \not\equiv \text{constant}$ and $K_u(x) = \nu K_v(x) \not\equiv \text{constant}$, where $\nu > 0$ in a nonempty open domain, then $(0, v^*(x))$ of (1) is not stable.

Proof. Consider the linearization of problem (1) and examine the associated eigenvalue problem of the equation for u

$$\nabla \cdot \left[a_1 \nabla \left(\frac{\phi}{K_u} \right) \right] + \phi f(x, 0, \nu v^*, K_u) = \sigma \phi, \quad x \in \omega, \quad \frac{\partial(\phi/K_u)}{\partial n} = 0, \quad x \in \partial\omega. \quad (17)$$

According to (Cantrell & Cosner, 2003), the principal eigenvalue is represented by

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \frac{- \int_{\omega} a_1 |\nabla(\phi/K_u)|^2 dx + \int_{\omega} \frac{\phi^2(x)}{K_u} f(x, 0, \nu v^*, K_u) dx}{\int_{\omega} (\phi^2/K_u) dx}$$

Upon replacing $\phi = K_u$ and using the property (e3) for f , we have

$$\sigma_1 \geq \frac{\int_{\omega} K_u F(x, \nu v^*, \nu K_v) dx}{\int_{\omega} K_u dx} = \frac{\int_{\omega} K_u F(x, v^*, K_v) dx}{\int_{\omega} K_u(x) dx}. \quad (18)$$

Thus $\sigma_1 > 0$ by using Lemma 2, which completes the proof. \square

Theorem 2. Let $P(x)$, $K_u(x) \not\equiv K_v(x) \not\equiv \text{constant}$ and the functions $g_1(x, u, v) = u f(x, u, K_u v/K_v, K_u)$ and $g_2(x, u, v) = v f(x, v, K_v u/K_u, K_v)$ satisfy (e2)-(e6). If $\frac{P(x)}{K_u(x)} \not\equiv \text{constant}$ and $K_u(x) = \nu K_v(x) \not\equiv \text{constant}$, for $\nu > 0$ on a nonempty open domain, then the equilibrium $(K_u(x), 0)$ of (1) is globally asymptotically stable. That is for any $u_0, v_0 \in C(\bar{\omega}) > 0$, the solution of (1) satisfies $(u, v) \rightarrow (K_u, 0)$ as $t \rightarrow \infty$.

Proof. Due to the following significant result, Lemma 4, Lemma 5, and Lemma 6 are valid for a strong monotone dynamical system. \square

Lemma 7. Let $P(x)$, $K_u(x)$, $K_v(x)$ are non-constant and the function f satisfies (e1)-(e6) as well as $P(x) \equiv K_u(x) \not\equiv \text{constant}$. If $K_u(x) > K_v(x)$ on $x \in \omega$, then $(0, K_v)$ for system (1) is unstable.

Proof. Taking the linearization of (1) around $(0, K_v)$ for $K_u > K_v$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot \left[a_1 \nabla \left(\frac{u}{K_u} \right) \right] + u f(x, 0, K_v, K_u), \quad t > 0, \quad x \in \omega, \\ \frac{\partial v}{\partial t} &= \nabla \cdot \left[a_2 \nabla \left(\frac{v}{P} \right) \right] + v f(x, K_v, 0, K_v) + K_v f_u(x, K_v, 0, K_v) u + K_v f_v(x, K_v, 0, K_v) v, \quad t > 0, \quad x \in \omega, \\ \frac{\partial(u/K_u)}{\partial n} &= \frac{\partial(v/P)}{\partial n} = 0, \quad x \in \partial\omega. \end{aligned}$$

and analyze the associated eigenvalue problem for u we get,

$$\nabla \cdot \left[a_1 \nabla \left(\frac{\phi}{K_u} \right) \right] + \phi f(x, 0, K_v, K_u) = \sigma \phi, \quad x \in \omega, \quad \frac{\partial(\phi/K_u)}{\partial n} = 0, \quad x \in \partial\omega. \quad (19)$$

Zahan, I. et al. (2022). A reaction-advection-diffusion model in spatial ecology: theoretical and computational analysis. *Khulna University Studies*, Special Issue (ICSTEM4IR): 203-218.

According to (Cantrell & Cosner, 2003), the principal eigenvalue is given by

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \frac{- \int_{\omega} a_1 |\nabla(\phi/K_u)|^2 dx + \int_{\omega} \frac{\phi^2(x)}{K_u} f(x, 0, K_v, K_u) dx}{\int_{\omega} (\phi^2/K_u) dx}$$

Upon replacing $\phi = K_u$ and using the property (e3) for f , we have

$$\sigma_1 \geq \frac{\int_{\omega} K_u F(x, K_v, K_u) dx}{\int_{\omega} K_u dx} = \frac{\int_{\omega} K_u F(x, v^*, K_u) dx}{\int_{\omega} K_u(x) dx}. \quad (20)$$

Thus $\sigma_1 > 0$ by using (5), which completes the proof. \square

Similarly under the assumption on Lemma (7) and for $K_v > K_u$ we can present the following remark.

Remark 1. Let $P(x)$, $K_u(x)$, $K_v(x)$ are non-constant and the function f satisfies (e1)-(e6) as well as $P(x) \equiv K_u(x) \not\equiv \text{constant}$. If $K_v(x) > K_u(x)$ on $x \in \omega$, then $(K_u, 0)$ for system (1) is unstable.

If both semi-trivial equilibrium are not stable then for a strongly monotone dynamical system, the model equation has at least one stable coexistence equilibrium (Matano, 1984). However, in our study Remark 1 and Lemma 7 indicate that the semi-trivial steady states $(K_u, 0)$ and $(0, K_v)$ of system (1) are unstable. Also, Lemma 4 is still holds for the system (1). Which reveals the following Theorem 3.

Theorem 3. Let $P(x) \equiv K_v(x) \not\equiv \text{constant}$ and $g_1(x, u, v) = uf(x, u, K_u v/K_v, K_u)$ and $g_2(x, u, v) = vf(x, v, K_v u/K_u, K_v)$ satisfy (e2)-(e5). If either $K_u(x) > K_v(x)$ or $K_u(x) < K_v(x)$ on a nonempty open domain ω , then (u_e, v_e) of (1) is globally stable.

Applications

In this part of the section, we will exhibit some numerical results both for the case of one and two dimensions in space to show the efficacy of the model that also completes the studied theoretical results of the earlier sections. For the case of a modified logistic function, the system (1) can be represented as

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot \left[a_1 \nabla \left(\frac{u}{K_u} \right) \right] + r_1 u \left(1 - \frac{u + K_u v/K_v}{K_u} \right), \quad t > 0, \quad x \in \omega, \\ \frac{\partial v}{\partial t} &= \nabla \cdot \left[a_2 \nabla \left(\frac{v}{P} \right) \right] + r_2 v \left(1 - \frac{K_v u/K_u + v}{K_v} \right), \quad t > 0, \quad x \in \omega, \\ \frac{\partial(u/K_u)}{\partial n} &= \frac{\partial(v/P)}{\partial n} = 0, \quad x \in \partial\omega. \end{aligned} \quad (21)$$

In this numerical computation we will study system (21) while considering either $P(x) \neq K_v$ or $P(x) \equiv K_v$ that will provides either competitive exclusion or coexistence of two competing species.

Case of one space dimension:

Example 1. Assume the model (21) with $a_1 = a_2 = 2.0$, $P = 1.5 + \sin(\pi x)$, $r_1 = r_2 = 1.0$ and $u_0 = v_0 = 0.5$ in Figure 1 for different unequal carrying capacity of the competing species where species v disperse in according to their resource function. The simulations informed that, the species which disperse choosing carrying capacity driven diffusion will survive in the battle which is also clear by Lemma 5 that present existence of no coexistence. However, by Theorem 2 with the increase of time all positive solutions approaches towards $(K_u(x), 0)$.

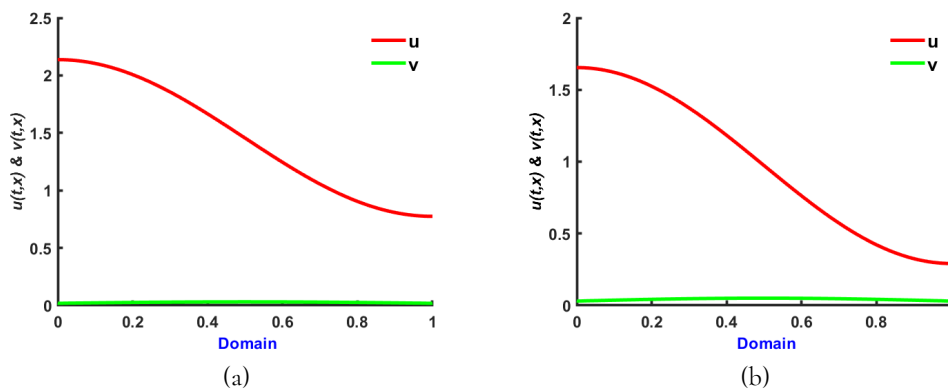


Figure 1: Population density profile Vs X of (21) for $a_1 = a_2 = 2.0$, $P = 1.5 + \sin(\pi x)$, $r_1 = r_2 = 1.0$ and $u_0 = v_0 = 0.5$ with (a) $K_u = (1.5 + 0.7 \cos(\pi x)) > K_v = (1.0 + 0.7 \cos(\pi x))$, and (b) $K_u = (1.0 + 0.7 \cos(\pi x)) < K_v = (1.5 + 0.7 \cos(\pi x))$ on $\omega = (0, 1)$.

Example 2. Assume the modified logistic growth for the model (21) for different identical and unequal carrying capacity of interacting species with $a_1 = a_2 = 2.0$, $r_1 = r_2 = 1.0$ and $u_0 = v_0 = 0.5$ where the growth rates of two rival species are constant. The simulation reported that, while both u and v following the same K – driven diffusion strategy then coexistence is possible to sustain for different deemed carrying capacity while the other parameters remain fixed. It is expected from Lemma 7 and the solution of the model (21) tends to (u_e, v_e) by Remark 1 and Theorem 3. However, for $K_u > K_v$ the population density of v is discovered marginally higher than u in Figure 2 (a) while opposite scenario is notice while considering $K_u < K_v$ in Figure 2 (c). Moreover, for $K_u = K_v$ the population density are noticed to merge with each other as shown in Figure 2 (b).

Zahan, I. et al. (2022). A reaction-advection-diffusion model in spatial ecology: theoretical and computational analysis. *Khulna University Studies*, Special Issue (ICSTEM4IR): 203-218.

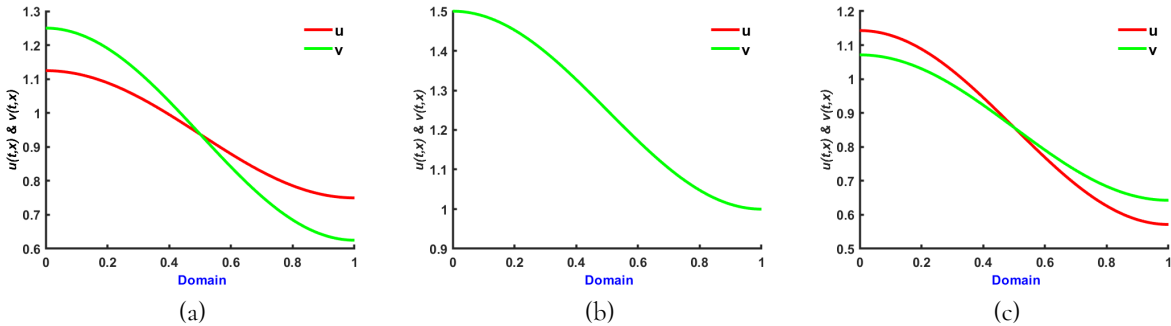


Figure 2: Population density profile Vs X of (21) for $a_1 = a_2 = 2.0$, $r_1 = r_2 = 1.0$ and $u_0 = v_0 = 0.5$ with (a) $K_u = (2.5 + 0.5 \cos(\pi x)) > K_v = P = (1.5 + 0.5 \cos(\pi x))$, (b) $K_u = K_v = P = 2.5 + 0.5 \cos(\pi x)$, and (c) $K_u = (1.5 + 0.5 \cos(\pi x)) < K_v = P = (2.0 + 0.5 \cos(\pi x))$ on $\omega = (0, 1)$.

Case of two space dimensions:

Example 3. Consider the problem (21) with $a_1 = a_2 = 2.0$, $K_u = (1.5 + \cos(\pi x) \cos(\pi y)) < K_v = (3.0 + \cos(\pi x) \cos(\pi y))$, $P = 2.0 + \sin(\pi x) \sin(\pi y)$, and $u_0 = v_0 = 0.5$ on $\omega = (0, 1) \times (0, 1)$ where K_u and K_v of two interacting species are non-constant and unequal. Figure 3 represents the surface profiles of u and v where letting $r_1 = 1.0 \gg r_2 = 0.01$ in Figure 3 (a) and $r_1 = 0.01 \ll r_2 = 1.0$ in Figure 3 (b). Here, u is dispersing based on their carrying capacity, but v is dispersing following their resource function. As stated by the Theorem 2, the species might experience competitive exclusion. However, we note that intrinsic growth rates provide a key contribution that demonstrates coexistence (see, Figure 3 (a), Figure 3 (b)). However, in both cases enhanced population density is detected in the surface profiles for letting elevated intrinsic growth rates.

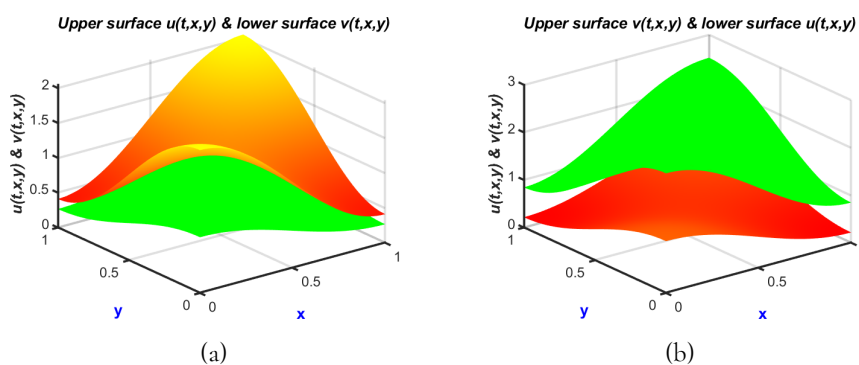


Figure 3: Equilibrium population densities of (21) for $a_1 = a_2 = 2.0$, $P = 2.0 + \sin(\pi x) \sin(\pi y)$, $K_u = (1.5 + \cos(\pi x) \cos(\pi y)) < K_v = (3.0 + \cos(\pi x) \cos(\pi y))$, and $u_0 = v_0 = 0.5$ with (a) $r_1 = 1.0 \gg r_2 = 0.01$, and (b) $r_1 = 0.01 \ll r_2 = 1.0$ on $\omega = (0, 1) \times (0, 1)$.

Example 4. Consider the case of (21) in Figure 4 for different unequal and non-constant carrying capacities of both interacting species where $a_1 = a_2 = 2.0$, $P = 1.5 + \sin(\pi x) \sin(\pi y)$, $r_1 = r_2 = 1.0$ and $u_0 = v_0 = 0.5$ on $\omega = (0, 1) \times (0, 1)$. In all circumstances, it is self-evident that one of the species will survive while the other will die out as $t \rightarrow \infty$. Furthermore, the contour profiles of species u mimic the trend of K_u where the highest values are noticed at the domain's left and right corners and the shape of the contour profile of species v follow the trend of P , where the optimum value is found at the middle of the contour domain. According to the model's theoretical analysis, when all other parameters are held constant, species that disseminate according to their individual carrying capacity will survive in completion.

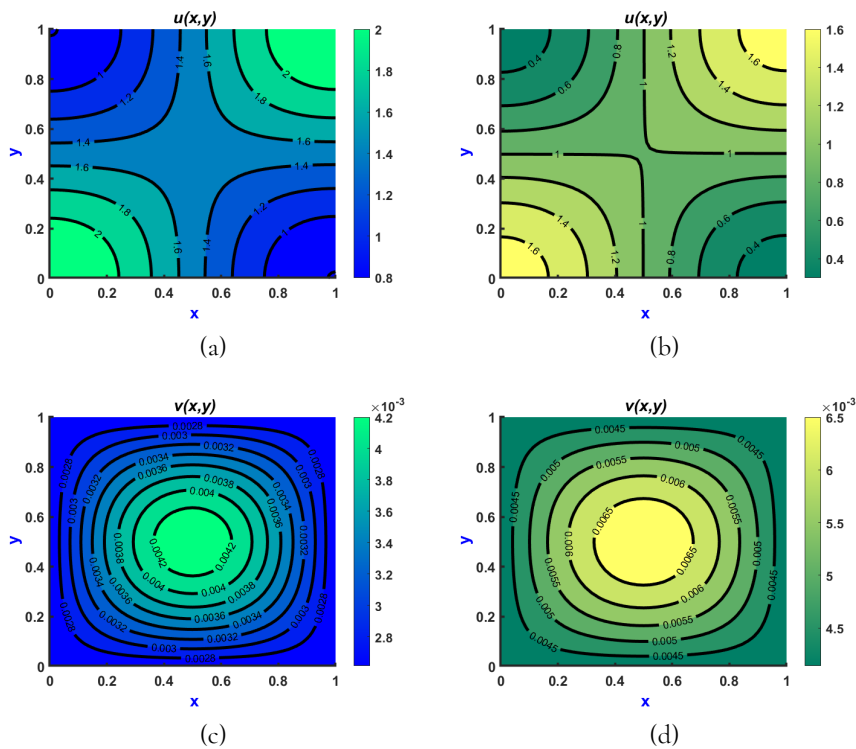


Figure 4: Contour plots of (21) for $a_1 = a_2 = 2.0$, $P = 1.5 + \sin(\pi x) \sin(\pi y)$, $r_1 = r_2 = 1.0$, and $u_0 = v_0 = 0.5$ with (a,c) $K_u = (1.5 + 0.7 \cos(\pi x) \cos(\pi y)) > K_v = (1.0 + 0.7 \cos(\pi x) \cos(\pi y))$ and (b,d) $K_u = (1.0 + 0.7 \cos(\pi x) \cos(\pi y)) < K_v = (1.5 + 0.7 \cos(\pi x) \cos(\pi y))$ on $\omega = (0, 1) \times (0, 1)$.

Example 5. Suppose the model of (21) while the diffusion strategy of two competing species is same with constant rate of advection. The computations are provided for a variety of non-constant carrying capacities, including $K_u = (2.0\pi e^{-(x-0.5)^2 - (y-0.5)^2} + 4.3) > K_v = P = (2.0\pi e^{-(x-0.5)^2 - (y-0.5)^2} + 1.2)$ in Figure 5(a,d), $K_u = K_v = P = 2.0\pi e^{-(x-0.5)^2 - (y-0.5)^2} + 1.2$ in Figure 5 (b,e) and $K_u = (2.0\pi e^{-(x-0.5)^2 - (y-0.5)^2} + 1.5) < K_v = P = (2.0\pi e^{-(x-0.5)^2 - (y-0.5)^2} + 3.5)$ in Figure 5 (c,f),

Zahan, I. et al. (2022). A reaction-advection-diffusion model in spatial ecology: theoretical and computational analysis. *Khulna University Studies*, Special Issue (ICSTEM4IR): 203-218.

respectively on $\omega = (0, 1) \times (0, 1)$. According to Theorem 3, no semi-trivial equilibrium should occur, and both species (u and v) should be able to survive mutually in the competition. It should also be noted that species with lesser carrying capacity can have slightly higher population density with constant advection rates (see, Figure 5 (a,d) and Figure 5 (c,f)). In-addition, given equivalent carrying capacities, both species' density levels coincide, as shown in Figure 5 (b,e).

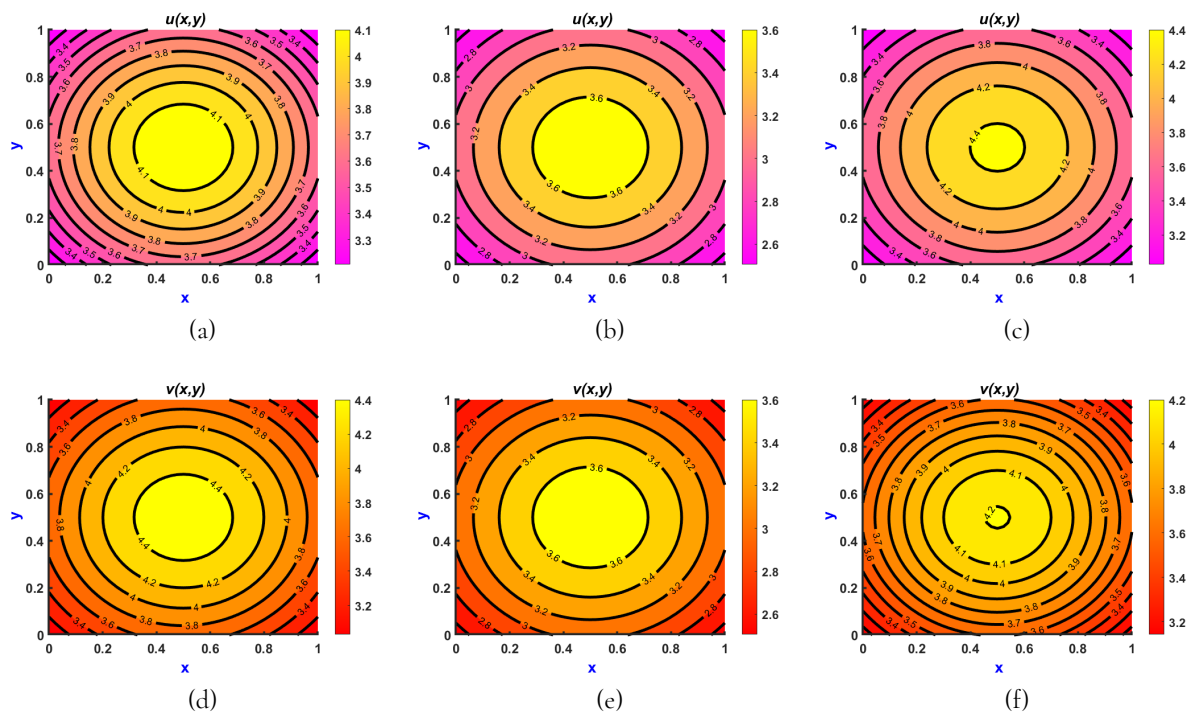


Figure 5: Contour plots of (21) for $a_1 = a_2 = 2.0$, $r_1 = r_2 = 1.0$ and $u_0 = v_0 = 0.5$ with (a,d) $K_u = (2.0\pi e^{-(x-0.5)^2-(y-0.5)^2} + 4.3) > K_v = P = (2.0\pi e^{-(x-0.5)^2-(y-0.5)^2} + 1.2)$, (b,e) $K_u = K_v = P = 2.0\pi e^{-(x-0.5)^2-(y-0.5)^2} + 1.2$, and (c,f) $K_u = (2.0\pi e^{-(x-0.5)^2-(y-0.5)^2} + 1.5) < K_v = P = (2.0\pi e^{-(x-0.5)^2-(y-0.5)^2} + 3.5)$ on $\omega = (0, 1) \times (0, 1)$.

Conclusions

In this study, we have considered two competing populations for generalized symmetric growth laws in a heterogeneous environment with different dispersal strategies which is in fact a very popular ecological model in adaptive dynamics. We have studied the model for two different circumstances where their carrying capacity is unequal and their growth function is symmetric. For different resource distribution of the competing species, while the resource function and the carrying capacity of the second species are not proportional then for $K_u(x) \equiv \nu K_v(x)$, $\nu > 0$ positive

solution should exist on their habitat. In this situation, the equilibrium $(u^*, 0)$ is stable globally where u^* tends to K_u as $t \rightarrow \infty$. However, for two interacting populations when they are following the identical strategy for their growth then for unequal resource distribution coexistence solution is found to sustain where elevated population density levels are the result of higher per capita growth rates. We have also studied numerically that, depending on the relation between the carrying capacities and the growth rates, coexistence or competitive exclusion should occur. The study can be extended for time dependent parameters when both populations follow the identical dispersal strategy with non-symmetric growth functions.

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