



DIFFERENT WAVELETS, THEIR PROPERTIES AND ADVANTAGES

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Abstract: In this article we discuss the concept of wavelets, different forms of wavelets and their properties, graphs of different wavelets and their Fourier transforms are shown. We have also discussed the comparative advantages of wavelet transforms over Fourier transforms- in analyzing signals.

Key words: Fourier transform, wavelet, wavelet transform, time-frequency signal analysis

Introduction

Wavelet analysis is an exciting new method for solving difficult problems in mathematics, physics and engineering, with modern applications in the fields of wave propagation, data compression, signal processing, image processing, pattern recognition, computer graphics and other medical image technology. Signal transmission is based on transmission of a series of numbers. The series representation of a function is important in all types of signal transmission. The wavelet representation of a function is a new technique. Wavelet transform of a function is the improved version of Fourier transform. Fourier transform is a powerful tool for analyzing the components of a stationary signal. But it is inadequate for analyzing the non stationary signals where-as wavelet transform allows the components of a non-stationary signal to be analyzed.

The first known connection to modern wavelets date back to Jean Baptiste Joseph Fourier in the nineteenth century. The next known link to wavelets came from Alfred Haar in the year 1909. After Haar's contribution to wavelets there was a gap of time in research in this field until a man named Paul Levy and slight advances were made in the field of wavelets from the 1930's to the 1970's. The next major advancement came from Jean Morlet around the year 1975. Morlet had made quite an impact on the history of wavelets; however, he wasn't satisfied with his efforts by any means. In 1981, Morlet teamed with man named Alex Grossman. Morlet and Grossman worked on an idea that

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Morlet discovered while experimenting on a basic calculator. The next two important contributors to the field of wavelets were Yves Meyer and Stephane Mallat, who applied the concept of wavelets in different applied problems. The introduced Multiresolution Analysis (MRA) for wavelets. The last wavelet researcher that we will mention is Ingrid Daubechies who made a great contributions in the wavelet theory.

Charles (1991), Christensen (2004), Daubechies (1992), Debnath (2002), Meyer (1993) extensively worked on wavelets. In this paper we have discussed about different wavelets, their properties and advantages.

Materials and Methods

Wavelet: Wavelet may be seen as a complement to classical Fourier decomposition method.

Suppose, a certain class of function is given and we want to find 'single valued (Simple) functions' f_0, f_1, f_3, \dots such that each

$$f(x) = \sum_{n=0}^{\infty} a_n f_n(x) \quad (1)$$

for some coefficients a_n .

Wavelet is a mathematical tool leading to representations of the type (1) for a large class of functions f .

Definition of wavelet: A wavelet means a small wave (the sinusoids used in Fourier analysis are big waves) and in brief, a wavelet is an oscillation that decays quickly.

Equivalent mathematical conditions for wavelet are:

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty; \quad (2)$$

$$\int_{-\infty}^{\infty} |\psi(t)| dt = 0; \quad (3)$$

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty; \quad (4)$$

where $\hat{\psi}(\omega)$ is the Fourier Transform of $\psi(t)$. Equation (4) is called the admissibility condition.

Wavelet transforms: Jean Morlet in 1982, introduced the idea of the wavelet transform and provided a new mathematical tool for seismic wave analysis. Morlet first considered wavelets as a family of functions constructed from translations and dilations of a single function called the "mother wavelet" $\psi(t)$. They are defined by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, a \neq 0 \quad (5)$$

The parameter a is the scaling parameter or scale, and it measures the degree of compression. The parameter b is the translation parameter which determines the time location of the wavelet. If $|a|$

< 1 , then the wavelet in (5) is the compressed version (smaller support in time- domain) of the mother wavelet and corresponds mainly to higher frequencies. On the other hand, when $|a| > 1$, then $\psi_{a,b}(t)$ has a larger time-width than $\psi(t)$ and corresponds to lower frequencies. Thus, wavelets have time-widths adapted to their frequencies. This is the main reason for the success of the Morlet wavelets in signal processing and time-frequency signal analysis.

Different wavelets and their properties: Haar wavelet: A function defined on the real line \mathfrak{R} as is known as the Haar wavelet.

$$\psi(t) = \begin{cases} 1 & \text{for } t \in \left[0, \frac{1}{2}\right) \\ -1 & \text{for } t \in \left[\frac{1}{2}, 1\right) \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

The Haar wavelet $\psi(t)$ is the simplest example of a wavelet. The Haar function $\psi(t)$ is a wavelet because it satisfies all the conditions of wavelet. Haar wavelet seems non smooth at $t = 0, \frac{1}{2}$ and discontinuous at $t = 1$ and it is very well localized in the time domain.

The Fourier transform of $\psi(t)$ is given by

$$\hat{\psi}(\omega) = i \exp\left(-\frac{i\omega}{2}\right) \frac{\sin^2\left(\frac{\omega}{4}\right)}{\frac{\omega}{4}}$$

$$\therefore \text{Re}\{\hat{\psi}(\omega)\} = \sin\left(\frac{\omega}{2}\right) \frac{\sin^2\left(\frac{\omega}{4}\right)}{\frac{\omega}{4}}$$

The graphs of Haar wavelet $\psi(t)$ and its Fourier transform $\hat{\psi}(\omega)$ are shown in Fig. 1(a) and Fig.1(b).

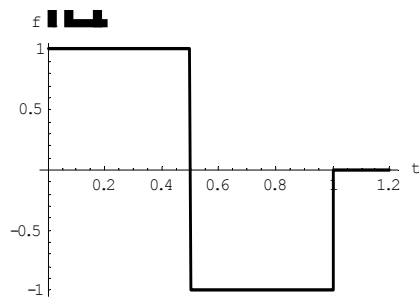


Fig. 1(a). Haar wavelet.

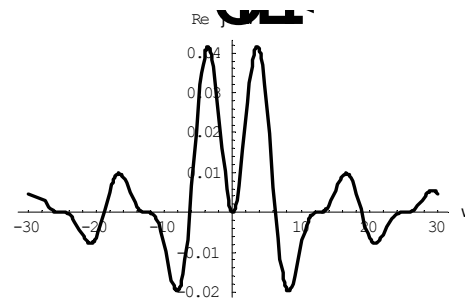


Fig. 1(b). Fourier transform of Haar wavelet.

Theorem: Haar function
$$\psi(t) = \begin{cases} 1 & \text{for } t \in \left[0, \frac{1}{2}\right) \\ -1 & \text{for } t \in \left[\frac{1}{2}, 1\right) \\ 0 & \text{otherwise} \end{cases}$$

is a wavelet because $\psi(t)$ satisfies the following conditions

$$\begin{aligned} \text{i)} \quad & \int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty; \\ \text{ii)} \quad & \int_{-\infty}^{\infty} \psi(t) dt = 0; \\ \text{iii)} \quad & \int_{-\infty}^{\infty} \frac{|\psi(\omega)|^2}{|\omega|} d\omega < \infty; \end{aligned}$$

Proof:

$$\begin{aligned} \text{i)} \quad \int_{-\infty}^{\infty} |\psi(t)|^2 dt &= \int_{-\infty}^0 |\psi(t)|^2 dt + \int_0^{\frac{1}{2}} |\psi(t)|^2 dt + \int_{\frac{1}{2}}^1 |\psi(t)|^2 dt + \int_1^{\infty} |\psi(t)|^2 dt \\ &= 0 + \int_0^{\frac{1}{2}} |1|^2 dt + \int_{\frac{1}{2}}^1 |-1|^2 dt + 0 = 1 \\ &\therefore \int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad \int_{-\infty}^{\infty} \psi(t) dt &= \int_{-\infty}^0 \psi(t) dt + \int_0^{\frac{1}{2}} \psi(t) dt + \int_{\frac{1}{2}}^1 \psi(t) dt + \int_1^{\infty} \psi(t) dt \\ &= 0 + \int_0^{\frac{1}{2}} (1) dt + \int_{\frac{1}{2}}^1 (-1) dt + 0 = 0 \\ &\therefore \int_{-\infty}^{\infty} \psi(t) dt = 0 \end{aligned}$$

iii) Evidently,

$$\begin{aligned} \hat{\psi}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} \psi(t) dt \\ &= \int_{-\infty}^0 e^{-i\omega t} \psi(t) dt + \int_0^{\frac{1}{2}} e^{-i\omega t} \psi(t) dt + \int_{\frac{1}{2}}^1 e^{-i\omega t} \psi(t) dt + \int_1^{\infty} e^{-i\omega t} \psi(t) dt \\ &= 0 + \int_0^{\frac{1}{2}} e^{-i\omega t} dt - \int_{\frac{1}{2}}^1 e^{-i\omega t} dt + 0 = i \exp\left(-\frac{i\omega}{2}\right) \frac{\sin^2\left(\frac{\omega}{4}\right)}{\frac{\omega}{4}} \end{aligned}$$

and $\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega = 16 \int_{-\infty}^{\infty} |\omega|^{-3} \left| \sin \frac{\omega}{4} \right|^4 d\omega < \infty$

We have shown that the Haar function $\psi(t)$ satisfies all the three conditions of wavelet.

Therefore, the Haar function $\psi(t)$ is a wavelet.

Mexican Hat wavelet: The wavelet which is defined by the second derivative of a Gaussian probability density function

$$\psi(t) = (1 - t^2) \exp\left(-\frac{t^2}{2}\right) = -\frac{d^2}{dt^2} \exp\left(-\frac{t^2}{2}\right) \tag{7}$$

is known as Mexican Hat Wavelet.

The Fourier transform of $\psi(t)$ is

$$\hat{\psi}(\omega) = \sqrt{2\pi} \omega^2 \exp\left(-\frac{\omega^2}{2}\right) \tag{8}$$

The graphs of Mexican Hat wavelet $\psi(t)$ and its Fourier transform $\hat{\psi}(\omega)$ are shown in Fig. 2(a) and 2(b). This wavelet has excellent localization in time and frequency domains and clearly satisfies the admissibility condition.

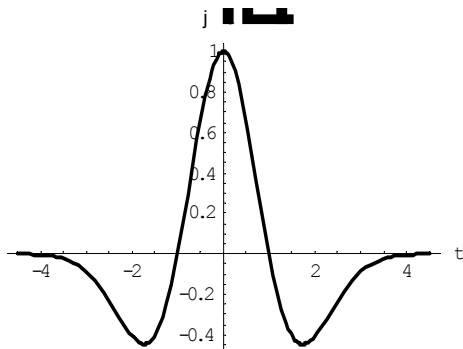


Fig. 2(a). The Mexican Hat wavelet.

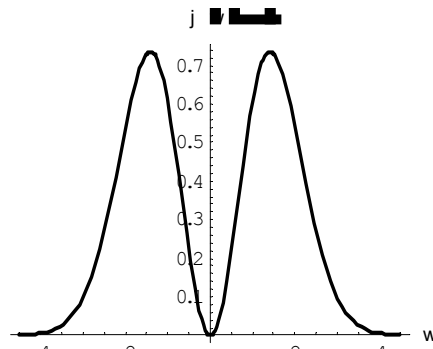


Fig. 2(b). Fourier transform of Mexican Hat wavelet.

Morlet wavelet: The wavelet which is defined by the Modulated Gaussian function is known as Morlet Wavelet.

$$\psi(t) = \exp\left(i\omega_0 t - \frac{t^2}{2}\right) \quad (9)$$

The Fourier transform of $\psi(t)$ is

$$\hat{\psi}(\omega) = \sqrt{2\pi} \exp\left[-\frac{1}{2}(\omega - \omega_0)^2\right] \quad (10)$$

The graphs of Morlet wavelet $\psi(t)$ and its Fourier transform $\hat{\psi}(\omega)$ are shown in Fig. 3(a) and 3(b). Morlet wavelet has compact support, regularity and continuity.

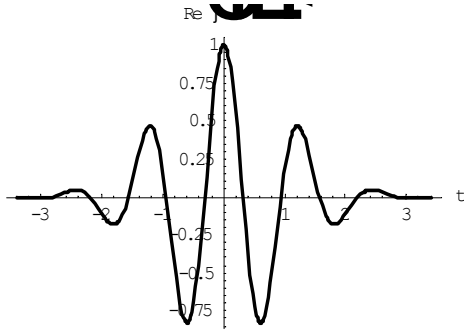


Fig. 3 (a). The Morlet wavelet.

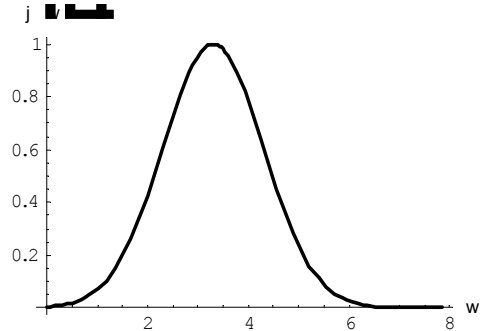


Fig. 3 (b). Fourier transform of Morlet wavelet.

Meyer wavelet: The Meyer wavelet and scaling function are defined in the frequency domain.

Shannon wavelet: Definition (Shannon function): The function $\psi(t)$ which is defined by -

$$\psi(t) = \frac{1}{\pi} (\sin 2\pi t - \sin \pi t) = \left(\frac{2}{\pi}\right) \sin\left(\frac{\pi}{2}\right) \cos\left(\frac{3\pi}{2}\right) \quad (11)$$

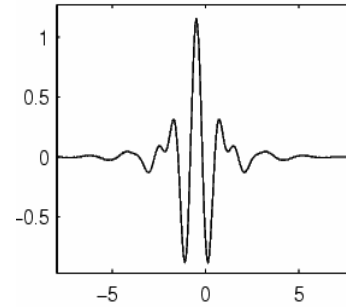
is known as the Shannon function.

The Shannon function $\psi(t)$ is the simplest example of a Shannon wavelet because the Shannon function satisfies all the conditions of wavelet.

The Fourier transform of $\psi(t)$ is given by

$$\hat{\psi}(\omega) = \begin{cases} 1, & \text{for } \pi < |\omega| < 2\pi \\ 0, & \text{otherwise} \end{cases} \quad (12)$$

The graphs of Shannon wavelet $\psi(t)$ and its Fourier transform $\hat{\psi}(\omega)$ are shown in Fig. 4(a) and 4(b).



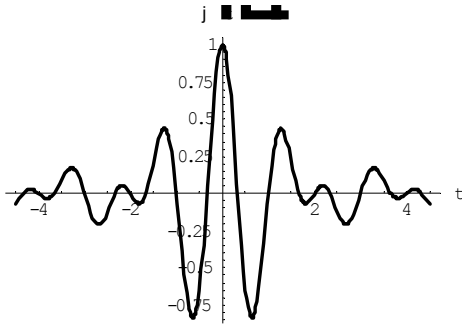


Fig. 4(a). The Shannon wavelet.

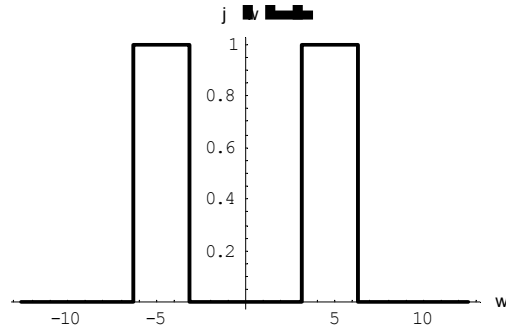


Fig. 4(b). Fourier transform of Shannon wavelet.

Fourier transform: Let $f \in L^1(\mathbb{R})$ The Fourier transform of a function $f(t)$ is denoted by $\hat{f}(\omega)$ and defined by

$$\hat{f}(\omega) = \Phi\{f(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt. \quad (13)$$

In Sometimes the Fourier transform is defined by Debnath (2002).

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i t \omega} f(t) dt$$

Frequently it is also convenient to use the notation

$$\hat{f}(\omega) = (\Phi f)(\omega);$$

Physically, the Fourier integral (13) measures oscillations of f at the frequency ω , and $\hat{f}(\omega)$ is called the frequency spectrum of a signal or waveform $f(t)$. It seems equally justified to refer to $f(t)$ as the waveform in the time domain and $\hat{f}(\omega)$ as the waveform in the frequency domain. Such terminology describes the duality and the equivalence of waveform representations.

Another modification is the definition without the minus sign in the kernel $\exp(-i\omega t)$. In electrical engineering, t and ω represent the time and the frequency respectively. In quantum physics and fluid mechanics, it is convenient to use the space variable x , the wave number k instead of t and ω respectively Christensen (2004), Debnath (2002).

Example 1 : Consider the *Rectangular Pulse* or *gate function*

$$R_r(t) = \begin{cases} 1 & \text{if } -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$$

The Fourier transform of this function is given by

$$\hat{R}_r(\omega) = \left(\frac{2}{\omega}\right) \sin(\tau\omega) \quad [\text{By using equation (13)}]$$

The graphs of $R_r(t)$ and its Fourier transform $\hat{R}_r(\omega)$ are shown in Fig. 5(a) and Fig. 5(b).

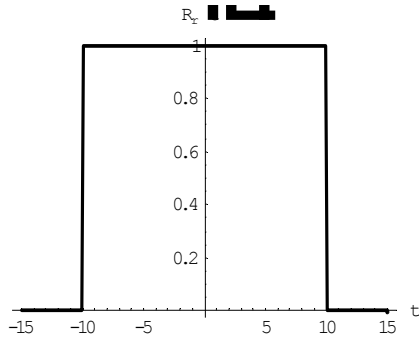


Fig. 5(a). Graphs of $R_r(t)$.

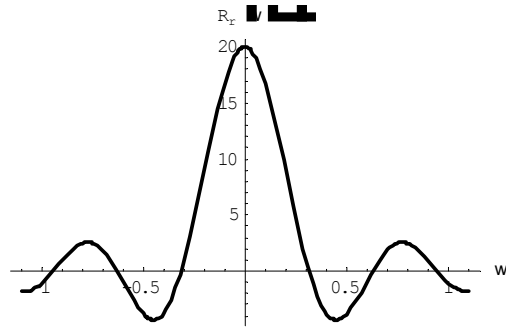


Fig. 5(b). Graphs of $\hat{R}_r(\omega)$

Example 2: Consider the *Triangular Pulse*

$$T_r(t) = \begin{cases} 1 + \frac{t}{\tau} & \text{if } -\tau \leq t < 0 \\ 1 - \frac{t}{\tau} & \text{if } 0 \leq t < \tau \\ 0 & \text{if } |t| > \tau \end{cases}$$

The Fourier transform of this function is given by

$$\hat{T}_r(\omega) = \tau \cdot \frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2} \quad [\text{By using equation (13)}]$$

The graphs of $T_r(t)$ and its Fourier transform $\hat{T}_r(\omega)$ are shown in Fig. 6(a) and Fig. 6(b).

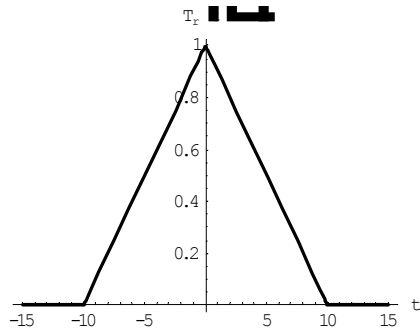


Fig. 6(a). Graphs of $T_r(t)$.

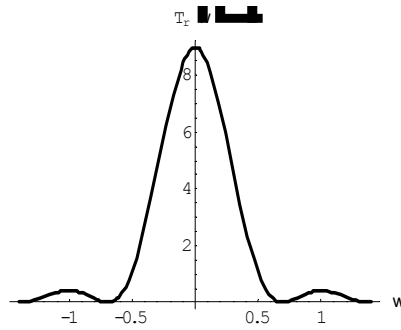


Fig. 6(b). Graphs of $\hat{T}_r(\omega)$.

A case for wavelet transform: Fourier analysis, using the Fourier transform, is a powerful tool for analyzing the components of a stationary signal (a stationary signal is a signal that repeats). For example, the Fourier transform is a powerful tool for processing signals that are composed of some combination of sine and cosine signals Mallat (1999).

The Fourier transform is less useful in analyzing non-stationary data, where there is no repetition within the region sampled. Wavelet transforms (of which there are, at least formally, an infinite

number) allow the components of a non-stationary signal to be analyzed. Wavelets also allow filters to be constructed for stationary and non-stationary signals Wells (1993), Strang (1989).

Initial wavelet applications involved signal processing and filtering. However, wavelets have been applied in many other areas including non-linear regression and compression. An offshoot of wavelet compression allows the amount of determinism in a time series to be estimated Walnut (2001), Wojtaszczyk (1997).

All types of signal transmission are based on transmission of a series of numbers. For signal transmission or signal storage the first step is to convert the given information to a series of numbers. To do this we need to represent a function f as a series representation. The function f is stored in the coefficients of the series and we can send only the coefficients. In practice we cannot send an infinite sequence of numbers. It is possible to send only a finite sequence of numbers. For good approximation usually this number forces to be large.

For series representation of a function, we consider a given function or signal f as

$$f(x) = \sum_0^{\infty} a_n f_n(x) \quad (14)$$

Where a_n 's are constant coefficients and f_0, f_1, f_2, \dots, K are single valued (or simple) functions.

The single valued (or simple) functions f_n may be polynomials or trigonometric functions.

For Taylor series approximation f should be analytic function.

For Fourier series approximation f should be periodic function. For details one can see Christensen, (2004).

Example: In signal analysis it is common to consider a function in $L_2(\mathbf{R})$.

i.e. $L_2(\mathbf{R}) = \left\{ \left\{ f : \mathbf{R} \rightarrow \mathbf{C} / \int_{\mathbf{R}} |f(x)|^2 < \infty \right\} \right\}$. This is never periodic except $f = 0$.

In that case we consider a wavelet function ψ such that $f(x) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} d_{j,k} \psi_{j,k}(x)$

Where $d_{j,k}$ are wavelet coefficients and $\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$ are the translated and scaled version of wavelet ψ .

For a periodic function the classical method is Fourier transform. But the main drawback of Fourier transform is that we lose our time information which is very important. The wavelet tool is the new tool to represent a function like (14). In the wavelet transform we do not lose the time information, which is useful in many contexts. Moreover there are many advantages of wavelet transform which are absent in Fourier transform. Some such types of advantages are mentioned below:

Advantages of wavelet transform: Wavelets offer a simultaneous localization in time and frequency domain. Wavelets separate the fine details in a signal. Very small wavelets can be used to isolate very fine details in a signal, while very large wavelets can identify coarse details. In wavelet theory, it is often possible to obtain a good approximation of the given function f by using only a few coefficients which is the great achievement in compare to Fourier transform. Most of the wavelet coefficients $\{d_{j,k}\}_{|j|,|k| \geq N}$ vanish for large N . Wavelet theory is capable of revealing aspects of data that other signal analysis techniques may miss like trends, breakdown points and discontinuities in higher derivatives and self-similarity. It can often compress or de-noise a signal without appreciable degradation.

Results

From the Figures of the wavelet transform and the Figures of the Fourier transform we observe that wavelets are well localized in both time and frequency domain whereas the standard Fourier transform is only localized in frequency domain. Fourier transform is based on a single function $\psi(t)$ and that this function is scaled. But for the wavelet transform we can also shift the function, thus generating a two-parameter family of functions $\psi_{a,b}(t)$ defined by (5).

Discussion

In Fourier analysis signal properties do not change over time and it is called a stationary signal. This draw back is not very important. But most interesting signals contain numerous non- stationary or transitory characteristics like drift, trends, abrupt changes and beginnings and ends of event. These characteristics are often the most important part of the signal. The classical Fourier analysis is not suited for detecting them but the wavelet analysis is suited for detecting them.

Conclusion

Different wavelet functions and their properties are discussed in this study. We have also tried to comparative discussion of Fourier transform and wavelet transform graphically with mentioning the drawback of Fourier transform. The advantages of wavelet transform are also focused. From our above discussion it is clear that wavelet transform is much more efficient than that of Fourier transform.

References

- Charles, C. 1991. *Wavelet Theory*. Academic Press, Cambridge, MA.
- Christensen, O. 2004. *Approximation Theory, From Taylor Polynomials to Wavelets*. Birkhäuser, Boston.
- Daubechies, I. 1992. *Ten Lectures on Wavelets*. SIAM, Philadelphia, PA.
- Debnath, L. 2002. *Wavelet Transformation and their Applications*. Birkhäuser, Boston.
- Mallat, S. 1999. *A wavelet Tour of Signal Processing*. Academi Press, New York.
- Meyer, Y. 1993. *Wavelets: their past and their future, Progress in Wavelet Analysis and its Applications*. Gif-sur-Yvette, pp. 9-18.
- Strang, G. 1989. Wavelets and Dilation Equations: A brief introduction. *SIAM Review*, 31: 614-627.
- Walnut, D.F. 2001. *An Introduction to Wavelet Analysis*, Birkhäuser, Boston.
- Wells, R.O. 1993. Parametrizing Smooth Compactly Supported Wavelets. *Transform American Mathematical Society*, 338(2): 919-931.
- Wojtaszczyk, P. 1997. *A Mathematical Introduction to Wavelets*. Cambridge University press, Cambridge, U.K.