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THE LATTICE OF ALL SEMI-SIMPLE CLASSES OF RINGS

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Abstract :The collection of all radicals of an associative ring is a lattice under the natural ordering of radicals. This lattice has been studied by Leavitt and Snider. In this paper it has been shown that the collection of all semi-simple classes of associative rings too is a lattice L_s under the natural ordering. The properties of this radical and its relation with the lattice of all radicals have been studied. A structure theorem has been established.

Key words: Lattice, Radical, Semi-simple

Introduction

Leavitt (1967) has shown that the class of all radicals of associative rings is a lattice under the natural ordering of inclusion. Snider (1972) has studied many properties of this lattice. We shall show that the class of all the semi-simple classes of associative rings too is a lattice under the natural ordering. We shall study a few properties of this lattice and its relation with the lattice of radicals. Some light has been shed on the structure of this lattice.

We have used the following notations:
L_R = the lattice of all radicals,
L_S = the lattice of all semi-simple classes,
L_C = the lower radical determined by C,
U_C = the upper radical determined by C,
S(R) = the class of all semi-simple rings with respect to the radical R,
R(A) = the R-radical of the ring A.

For terminology and basic results, see Divinsky (1965) and Wiegandt (1983), Gratzer (1971), Rutherford (1965). All rings considered here are associative.

We recall that a non-empty class C of rings is a semi-simple class if C is the class of all semi-simple rings with respect to some radical R. We also recall that if R_1 and R_2 are two radicals, then $R_1 \wedge R_2 = R_1 \cap R_2$ and $R_1 \vee R_2 = L_{R_1 \cup R_2}$

We first prove

Theorem 2.1

Let $\{S_\alpha\}$ be a non-empty collection of semi-simple classes. Then $\cap_\alpha S_\alpha$ is a semi-simple class.

Proof

Let $S_\alpha = S(R_\alpha)$. Then $A \in \cap_\alpha S_\alpha$ if and only if, for each α, A has no non-zero R_α-ideals, i.e., if and only if A has no $(\cup_\alpha R_\alpha)$- ideals, i.e., if and only if A is $L_{(\cup_\alpha R_\alpha)}$- semi-simple i.e., if and only if $A \in S(L_{(\cup_\alpha R_\alpha)})$.

Hence $\cap_\alpha S_\alpha = (L_{(\cup_\alpha R_\alpha)})$

We shall now consider the natural ordering in L_S. For $S_1, S_2 \in L_S$, $S_1 \leq S_2$ if and only if $S_1 \subseteq S_2$. It is easily seen that the infimum $S_1 \wedge S_2$ and supremum $S_1 \vee S_2$ of S1and S_2 are given by

$$S_1 \wedge S_2 = S_1 \cap S_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$S_1 \vee S_2 = \bigcap_{\substack{S \in L_S \\ S_1 \vee S_2}} S \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Theorem 2.1 therefore follows.

Theorem 2.2

L_S is a lattice* under natural ordering of inclusion.

Our next result will establish completeness of the lattice L_S. Before its proof we recall that S is a semi-simple

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class if and only if S is the class of all of rings A every non-zero ideal of which can be mapped homomorphically onto a non-zero ring in S.

Theorem 2.3

(i) $\wedge_\alpha S_\alpha = S(\vee_\alpha R_\alpha)$, where $S_\alpha = S(R_\alpha)$

(ii) $\vee_\alpha S_\alpha = S(U_{(\cup_\alpha S_\alpha)})$

Proof

(i) follows from theorem 2.1.

Now $A \in S(U_{(\cup_\alpha S_\alpha)})$ if and only if every non-zero ideal of $\cup_\alpha S_\alpha$ can be mapped homomorphically onto a non-zero ring in $\cup_\alpha S_\alpha$. Thus $S(U_{(\cup_\alpha S_\alpha)})$ is the semi-simple closure of $\cup_\alpha S_\alpha$, i.e. the smallest semi-simple class containing $\cup_\alpha S_\alpha$. Hence by theorem 2.1 and definition of '∨', $\vee_\alpha S_\alpha = S(U_{(\cup_\alpha S_\alpha)})$.

We consequently have

Corollary 2.4.

L_S is complete.

*It is well known that L_S is no a set, as is not L_R (Snider, 1972).

Since U, the class of all rings and O, the class consisting of the only ring O are semi-simple classes, L_S is a bounded lattice.

We now prove

Theorem 2.4

L_S is dual atomic.

Proof

Let $\overline{S}_0 = S(L_{\{S_0\}}$, where S_0 a simple ring. Then $\overline{S}_0$ is a dual atom in L_S. Let $S \in L_S$ such that $\overline{S}_0 <$ S, and let S = S(R). Then there is a ring A which is R- semi-simple but not $L_{\{So\}}$ semi-simple. Hence $L_{\{So\}}(A)$ is R-semi-simple but $L_{\{So\}}$- radical. Since all $L_{\{So\}}$semi-simple rings (i.e. $\overline{S}_0$-rings) are R-semi-simple (i.e. S-rings), $R < L_{\{So\}}$, Snider (1972) has proved that $L_{\{So\}}$ is an atom in L_R. Hence R = 0, and so, S = U $\overline{S}_0$ is therefore a dual atom in L_S

We shall now prove a theorem that will throw light on the structure of L_S. Let Σ denote the class of simple rings, and let B(Σ) denote the Boolean algebra of all subclasses of Σ.

Then we have,

Theorem 2.5

The map $f: L_S \to B(\Sigma)$ given by $f(S) = \Sigma_S$, where Σ_S is the class of all simple rings in S, is a lattice homomorphism onto B(Σ).

Proof

Let $S_1, S_2 \in L_S$. Then, $\sum_{S_1} \cap \sum_{S_2} = \sum_{S_1 \cap S_2} = \sum_{S_1 \wedge S_2}$, and so $f(S_1 \wedge S_2) = f(S_1) \cap f(S_2)$. Now let $A \in \Sigma$. Since A is simple, $A \in S_1 \vee S_2 = S(U_{(S_1 \cup S_2)})$ if and only if A can be mapped homomorphically onto a non-zero ring in $S_1 \cup S_2$ i.e. if and only if $A \in S_1 \cup S_2$, i.e. if and only if $A \in \sum_{S_1} \cup \sum_{S_2}$. Hence $f(S_1 \vee S_2) = f(S_1) \cup f(S_2)$. f is obviously onto. The proof is thus complete.

We conclude with a theorem connecting L_S with L_R^{op}, the opposite lattice of L_R. We denote the join and the meet in L_R^{op} by ∨′ and ∧′. Thus, if $R1, R_2 \in L_R^{op}$,
$R_1 \vee' R_2 = R_1 \wedge R_2$ and $R_1 \wedge' R_2 = R_1 \vee R_2$

Theorem 2.6

L_S is lattice isomorphic to L_R^{op}.

Proof

Define $\varphi: L_S \rightarrow L_R^{op}$ by $\varphi(S) = R$, where $S = S(R)$.

Since $S_1 \leq S_2$ implies $R_2 \leq R_1$, where $S_1 = S(R_1)$, $S_2 = S(R_2)$, φ is order preserving. The 1 – 1 and onto properties of φ are obvious.

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